APR 9 - 1929

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AMERICAN JOURNAL OF MATHEMATICS

FOUNDED BY THE JOHNS HOPKINS UNIVERSITY

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PUBLISHED UNDER THE JOINT AUSPICES OF

THE JOHNS HOPKINS UNIVERSITY
AND
THE AMERICAN MATHEMATICAL SOCIETY

Volume LI, Number 2

APRIL, 1929

THE JOHNS HOPKINS PRESS
BALTIMORE, MARYLAND
U. S. A.

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THE AMERICAN JOURNAL OF MATHEMATICS will appear four times yearly.

The subscription price of the JOURNAL for the current volume is $7.50 (foreign postage 25 cents); single numbers $2.00.

A few complete sets of the JOURNAL remain on sale.

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Subscriptions to the JOURNAL and all business communications should be sent to THE JOHNS HOPKINS PRESS, BALTIMORE, MARYLAND, U. S. A.

Entered as second-class matter at the Baltimore, Maryland, Postoffice, acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, Authorized on July 3, 1918

PRINTED BY THE J. H. FURST COMPANY
BALTIMORE, MD.

Singular Points of Vector Fields under General Boundary Conditions.*

By Marston Morse.

I.

1. *Introduction.* In his work on differential equations Poincaré † found it necessary to study the distribution of singular points of vector fields. Following Poincaré this study has been continued by many.‡ Recently Hopf § and Lefschetz ¶ have made notable additions. The work of Hopf is based on Brouwer's ‖ fundamental notion of the degree of a transformation.

The theory as developed centers around the sum of the indices of the singularities. Up to the present paper this sum has been determined on general manifolds only for the special case where the vectors on the boundary are all interiorly or exteriorly directed. The evaluation for this case has been obtained both by Hopf and Lefschetz.

The principal advance of the present paper is the evaluation of this index-sum under general boundary conditions. The result obtained was suggested by an analogous problem in the study of critical points of functions.** The index-sum turns out to be a topological invariant not of the complex C_n on which the field is defined, but of a sequence of $n+1$ complexes associated

* Presented at the summer meeting of the Society at Amherst on September 6, 1928.

† Poincaré, "Sur les Courbes Définies par une Équation Differentielle," *Journal de Mathématiques*, Ser. 3, Vol. 7 (1881), pp. 375-421.

‡ For the older references see von Kerékjarto, *Vorlesungen über Topologie*, I, pp. 265-266.

§ Hopf I, "Curvatura integra geschlossener Hyperflächen," *Mathematische Annalen*, Vol. 95 (1925), pp. 340-351; Hopf II, "Abbildungsklassen n-dimensionaler Mannigfaltigkeiten," *Mathematische Annalen*, Vol. 96 (1296), pp. 209-224; Hopf III, "Vectorfelder in n-dimensionalen Mannighaltigkeiten," *Mathematische Annalen*, Vol. 96 (1926), pp. 225-249.

¶ Lefschetz, "Manifolds with a Boundary and Their Transformations," *Transactions of the American Mathematical Society*, Vol. 29 (1927), pp. 429-463.

‖ Brouwer, "Über Abbildung von Mannigfaltigkeiten," *Mathematische Annalen*, Vol. 71 (1911), pp. 97-115.

** Morse, "The Analysis and Analysis Situs of Regular n-Spreads in $(n+r)$-Space," *Proceedings of the National Academy of Sciences*, Vol. 13, pp. 813-817. Here an abstract is given of results to be published in full later.

with the field, a *sequence characterizing the field* as it will be called. A minor advance consists in giving a new definition of the index of infinite sets of singularities. This definition does not rest simply on an existence theorem, but is in a form immediately applicable to problems in analysis, for example, to the case where one has a regular curve of singular points.

2. *The complexes. Let $\sum_n$ be a closed manifold in the sense of Analysis Situs.* We suppose also that $\sum_n$ possesses a Riemannian metric,†* that is, that every sufficiently small neighborhood of each point of $\sum_n$ is homeomorphic to the neighborhood of a point in an Euclidian space of coördinates $(x)=(x_1, \cdots, x_n)$ and thereby has a metric defined by a positive definite quadratic form,

$$ds^2 = \sum_{ij} g_{ij}\, dx_i\, dx_j \qquad (i, j = 1, 2, \cdots, n).$$

We shall not assume that the g_{ij}'s are analytic in (x), but simply of class C'', that is, continuously differentiable up to the second order. The restriction that $\sum_n$ have a Riemannian metric could be removed by the substitution of suitable abstractions, but the writer is unwilling to sacrifice brevity, transparency, and applicability for the slight advantage thereby to be gained. Manifolds with a Riemannian metric include all the ordinary r-manifolds in Euclidian $(n+r)$-space, as well as the so-called characteristic manifolds of dynamical systems.

Let $(u)=(u_1, \cdots, u_r)$ be a set of rectangular coördinates in an auxiliary Euclidian r-space. By an *r-element* on $\sum_n$, $o < r \leqq n$, will be meant a set of points on $\sum_n$ homeomorphic with an r-simplex in the space (u). An r-element will be called *regular* if on the element the Riemannian coördinates (x) of the preceding section are functions of the u's of at least class C'', and if at least one set of r of the variables (x) has a jacobian with respect to the u's which is not zero. A *closed r-manifold* on $\sum_n$ will be called *regular* if each of its points possesses a neighborhood which is a regular r-element. An *r-manifold with boundary B_{r-1}* will be called *regular* if it is a connected subcomplex of a regular closed r-manifold, and if its boundary B_{r-1} consists of a finite set of distinct, regular, closed $(r-1)$-manifolds. An *r-complex* will be called *regular* if it consists of a finite set of distinct, regular r-manifolds, either with or without boundary.

3. *The vector field.* Let there be given on $\sum_n$ any regular n-manifold

* Veblen, "Analysis Situs," *The Cambridge Colloquium*, Part II.

† Eisenhart, *Riemannian Geometry*.

C_n with boundary B_{n-1}. On C_n let there be given a continuous contravariant vector field H_n. See Eisenhart *loc. cit.* We shall designate the vectors of the field H_n also by H_n, and the components of H_n by $[h_1, h_2, \cdots, h_n]$. A point on C_n at which all of the components of H_n are zero is called a singular point. Our principal assumptions may be summarized as follows:

Hypotheses. We assume that we have a regular n-manifold C_n with boundary B_{n-1} on a closed n-manifold $\sum_n$ with a Riemannian metric. On C_n we have a contravariant vector field H_n, continuous on C_n, and non-singular on B_{n-1}.

To proceed further we must define the index of a singularity of the vector field. See Brouwer, and Hopf I, *loc. cit.*

4. *The index of isolated singularities.* In the spherical neighborhood of a point P of Euclidian n-space let there be given a continuous vector field H_n with an isolated singularity at P. Let J_{n-1} be a Jordan $(n-1)$-manifold that contains P in its interior, and that lies wholly in the given neighborhood. Let S_{n-1} be a unit $(n-1)$-sphere in the Euclidian n-space. Let J_{n-1} and S_{n-1} be oriented so that their orders with respect to interior points are either both 1 or -1.*

We understand that the given vector field H_n *defines a transformation* F of J_{n-1} onto S_{n-1} in which a point Q on J_{n-1} corresponds to that point on S_{n-1} at which S_{n-1} is met by a ray from its center parallel to the vector of H_n at Q. We shall call F the *index determining transformation* and define the *index* v of the singularity of H_n at P as the degree of F. (Abbildungsgrad, see Brouwer first reference.)

From the properties of v considered as the degree of F it follows that v is unchanged by any continuous deformation of H_n in which H_n remains non-singular on J_{n-1}. According to Hopf I, *loc. cit.*, v is the same for any other Jordan manifold containing P; it is independent of the orientation of J_{n-1}, provided S_{n-1} be similarly oriented; it is independent of any continuously differentiable transformation of the neighborhood of P, provided such a transformation have a non-vanishing Jacobian, and provided we understand the vectors H_n transformed as contravariant vectors in Riemannian Geometry, that is in the same manner as the differentials of the coördinates.

We return to the manifold C_n of § 3 and its vector field H_n. Let P be an isolated singular point of H_n. Suppose that P lies interior to a regular

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* Brouwer, "Über Jordansche Mannigfaltigkeiten," *Mathematische Annalen*, Vol. 71 (1912), p. 323.

element of coördinates (x). In the Euclidian space of the coördinates (x) the components of H_n will define a vector field $\overline{H}_n$. Suppose P has the coördinates (x^0). We will define the *index of P on C_n* as the index of $\overline{H}_n$ at (x^0). This index will be independent of the coördinates (x) used to regularly represent C_n in the neighborhood of P, since any other set of such coördinates can be obtained from the coördinates (x) by a continuously differentiable transformation with a non-vanishing Jacobian.

A complex will be said to be *connected with respect to n-cells* if any two of its points lie on a finite ordered set of n-cells of which each has at least one $(n-1)$-cell in common with its predecessor. From the work of Hopf II, *loc. cit.*, we can obtain the two following lemmas.

LEMMA A. *Let there be given a complex D_n that is connected with respect to n-cells and that has a boundary D_{n-1}. On D_n let H_n be a continuous vector field with no singular points on D_{n-1}. Without changing D_n on D_{n-1}, H_n can be continuously deformed on D_n into a field K_n that has just one singular point P.*

LEMMA B. *If the field H_n of Lemma A is one of isolated singular points of index sum v, and if D_n is orientable, the final field K_n in the neighborhood of P may be taken as any field that has a singularity of index v at P.*

The theorem proved by Hopf and Lefschetz is the following:

THEOREM 1. *If a field H_n on C_n has isolated singularities, and is interiorly directed at each point of the boundary of C_n then the index sum of the singularities of H_n equals $(-1)^n$ times the Euler-Poincaré characteristic of C_n.*

5. *The case where the vectors which are exteriorly normal to the boundary are isolated.* Let there be given on $\sum_n$ a manifold C_n with boundary B_{n-1}, and on C_n a field H_n satisfying the hypotheses of § 3. We shall treat first the case where there are most a finite number of points, say, P_k, $k = 0, 1, \cdots, r$, at which H_n is exteriorly normal to B_{n-1}. To proceed further we need to define the projection of H_n on B_{n-1}.

We refer the part of $\sum_n$ neighboring any point P of B_{n-1} to geodesic coördinates. That is, we take regular parameters $(x_1, \cdots, x_n)$ of which x_n represents the distance measured from B_{n-1} along a geodesic g normal to B_{n-1}, and x_n is taken so as to be positive outside of C_n. On B_{n-1} the first $n-1$ components of H_n will define a vector field H_{n-1} that will be contravariant on B_{n-1}. We term H_{n-1} *the projection of H_n on H_{n-1}.*

The field H_{n-1} will have isolated singularities at the above points P_k. We denote the index of H_{n-1} at P_k by v_k. For any two sets of our geodesic coördinates the component h_n of H_n at any point of B_{n-1} is the same function of the point on B_{n-1}. For h_n is readily seen to be the product of the length of H_n and the cosine of the angle at P between H_n and the geodesic g, using the terms length and angle in the senses of Riemannian Geometry. The component h_n on B_{n-1} is thereby proved continuous, and is accordingly less in absolute value than some positive constant M. We shall now prove the following lemma:

LEMMA. *It is possible to extend the definition of the field H_n beyond the boundary B_{n-1} of C_n to a geodesic manifold $x_n = e$, where e is a sufficiently small positive constant, in such a manner that the vectors H_n of the extended field are all directed interiorly on the new boundary, and the extended field has no new singularities other than isolated singularities Q_k of index $-v_k$, respectively, on the geodesics g_k exteriorly normal to B_{n-1} at the points P_k.*

Let e be an arbitrarily small positive constant. Our field H_n will now be extended as follows. For the points $(x_1, \cdots, x_n)$ for which

$$0 < x_n \leqq e \tag{1}$$

the first $(n-1)$ components of H_n will be taken as equal to those of H_n at $(x_1, \cdots, x_{n-1}, 0)$, while for the same points we set

$$h_n(x_1, \cdots, x_n) = h_n(x_1, \cdots, x_{n-1}\ 0) - x_n(M/e). \tag{2}$$

The extended field H_n thereby defined is seen to have, beside the original singularities of H_n, an isolated singular point, Q_k, lying between B_{n-1} and the new boundary $x_n = e$, on the geodesic g_k normal to B_{n-1} at P_k.

It remains to show that the index of Q_k is $-v_k$.

To determine this index of H_n at Q_k we note that Q_k is interior to the region satisfying (1), and the following condition,

$$(x_1 - a_1)^2 + \cdots + (x_{n-1} - a_{n-1})^2 \leqq r^2, \tag{3}$$

where $(a_1, \cdots, a_{n-1}, 0)$ represents the point P_k and where r is so small a positive constant that h_n is positive on B_{n-1} throughout (3). Let Z_{n-1} be the boundary of this region. The index of Q_k is the degree v of that transformation F of Z_{n-1} on a unit $(n-1)$ sphere R_{n-1} which is defined by the vectors H_n on Z_{n-1}. This degree v will be unaltered if we subject H_n on Z_{n-1} to the two following continuous deformations, D_1 and D_2, thought of as made in the space (x).

The first deformation D_1 is one that can be affirmed to be possible by virtue of Lemmas A and B of § 4. In it we deform the vectors on Z_{n-1} at which $x_n = 0$, and only these, and for these vectors we leave the component h_n unchanged. The remaining components form a vector field, H_{n-1} on $x_n = 0$, within and on the $(n-2)$-sphere S_{n-2}, defined by the equality in (3). Without altering H_{n-1} on S_{n-2} we continuously deform H_{n-1} interior to S_{n-2} into a field K_{n-1} with the following properties. Its only singularity is to be at P_k. On the sphere S'_{n-2} concentric with S_{n-2} and with radius $r/2$, the vectors K_{n-1} are to be of length equal to $r/2$ and such that K_{n-1} defines a transformation of S'_{n-2} onto a similarly oriented unit $(n-2)$-sphere $S_{n-2}{}^0$, into a Riemann hypersurface of $|v_k|$ leaves on $S_{n-2}{}^0$, by a transformation whose degree is v_k. On each sphere concentric with S_{n-2} and of radius ρ less than $r/2$, the field is to be similar to that on S_{n-2} but with its vectors all of length ρ.

The second deformation D_2 is one in which we continuously change h_n on Z_{n-1} so as to make h_n a positive constant in a neighborhood N of P_k, instead of simply positive. This can be effected without altering h_n outside of a neighborhood of P_k slightly larger than N.

The vector field H'_n on Z_{n-1}, resulting from the deformations D_1 and D_2 defines a transformation F', of Z_{n-1} on a unit $(n-1)$-sphere R_{n-1}, whose index is the required index of Q_k. Under F' the point P_k goes into some point P on R_{n-1}, and the neighborhood of P_k on Z_{n-1} goes into a neighborhood of P on R_{n-1} covered $|v_k|$ times. Further, under F' no other point of Z_{n-1} goes into a sufficiently small sub-neighborhood of P, since no other point of Z_{n-1} besides P_k bears a vector of H_n parallel to that at P_k. Finally from the fact that the vector at P_k on Z_{n-1} is interiorly normal to Z_{n-1}, and from the conventions regarding orientations, it follows that the desired index of $Q_k = -v_k$. Thus the lemma is proved.

Suppose in the present section that the singularities of H_n are isolated. The extended field H_n will have isolated singularities and be interiorly directed on the boundary of the extended domain C'_n. According to Theorem 1 the index-sum of the singularities of H_n on C_n equals (-1) times the Euler-Poincaré characteristic of C_n. Since the characteristic of C_n is clearly that of C'_n we have the following theorems.

THEOREM A_0. *If H_n is exteriorly normal to B_{n-1} at most at isolated points, then the index-sum I_n, of H_n, is given as follows:*

$$I_n = (-1)^n E_n + I_{n-1} \qquad n > 0 \tag{4}$$

where E_n is the characteristic of C_n, and I_{n-1} is the index-sum of the projection on B_{n-1} of the exteriorly directed vectors of H_n.

The theorem holds for $n=1$, if we agree that a vector at a boundary point a_0 of a one-dimensional manifold is normal to a_0, and projects on a_0 into a null vector of index 1.

On B_{n-1} let C_{n-1} be any regular complex with boundary B_{n-2}, that contains in its interior all of the points at which H_n is exteriorly normal to B_{n-1}, and contains none of points at which H_n is interiorly normal to B_{n-1}. Let H_{n-1} be the projection of H_n on C_{n-1}. Let E_{n-1} be the characteristic of C_{n-1}. We come now to the following theorem.

THEOREM A_1. *If the fields H_n and H_{n-1}, respectively, are exteriorly normal to B_{n-1} and B_{n-2} at most at isolated points, then*

$$I_n=(-1)^n(E_n-E_{n-1})+I_{n-2}, \tag{5}$$

where I_{n-2} is the index-sum of the projection on B_{n-2} of the exteriorly directed vectors of H_{n-1}.

To prove this theorem we note first from (4) that

$$I_{n-1}=(-1)^{n-1}E_{n-1}+I_{n-2}, \tag{6}$$

and then use (4) and (6) to get (5).

The definitions preceding Theorem A_1 correspond to the case $i=1$ of the following definitions, given successively for $i=1, 2, \cdots, n$.

On B_{n-i} let C_{n-i} be a regular complex, with boundary B_{n-i-1} (possibly null), that contains (in its interior if $n \neq i$) all of the points at which H_{n-i+1} is exteriorly normal to B_{n-i}, and contains none of the points at which H_{n-i+1} is interiorly normal to B_{n-i}. Let H_{n-i} be the projection of H_{n-i+1} on C_{n-i}. Let E_{n-i} be the characteristic of C_{n-i}.

We come now to a succession of theorems of which Theorem A_1 is the first. Here $i=1, 2, \cdots, n-1$

THEOREM A_i. *If the fields $H_n \cdots H_{n-i}$ are exteriorly normal to $B_{n-1} \cdots B_{n-i-1}$, respectively, at most at isolated points, then*

$$I_n=(-1)^n[E_n-E_{n-1}+\cdots+(-1)^nE_{n-i}]+I_{n-i-1},$$

where I_{n-i-1} is the index-sum of the projection on B_{n-i-1} of the exteriorly directed vectors of H_{n-i}.

If in particular H_{n-k} is never exteriorly normal to B_{n-k-1}, then I_{n-k-1} is zero, and C_j may be taken as a null complex for each $j<n-k$. Each Theorem A_i for which $i \geqq k$ will then give

$$I_n = (-1)^n \left[E_n - E_{n-1} + \cdots + (-1)^k E_{n-k}\right]$$

In Theorem A_{n-1} we note that I_0 must reduce to E_0 according to our conventions. We have then the following theorem.

THEOREM 2. *If the fields $H_n \cdots H_1$ are exteriorly normal respectively to $B_{n-1} \cdots B_0$ at most at isolated points, then*

$$I_n = E_0 - E_1 + \cdots + (-1)^n E_n,$$

where E_i is the characteristic of C_i, and I_n is the index-sum of H_n.

6. *General Boundary conditions.* The preceding hypothesis that the various fields of projections have isolated singularities will now be removed.

When a field H_r is defined over a complex C_r, H_r and C_r will be called a *corresponding* field and complex. In the preceding section the complexes C_r and fields H_r have the following properties.

They form a sequence of regular complexes

$$C_n C_{n-1} \cdots C_0, \tag{1}$$

and corresponding fields,

$$H_n H_{n-1} \cdots H_0, \tag{2}$$

of which each complex is on the boundary of its predecessor, and each field H_r is the projection of the preceding field on the complex C_r corresponding to H_r, while each complex is so chosen on the boundary of the preceding complex as to contain none of the points at which the preceding field is interiorly normal, and to contain in its interior all of the points on the boundary of the preceding complex at which the preceding field is exteriorly normal.

A sequence of complexes (1) with the properties of the preceding paragraph will be called a sequence of complexes characterizing H_n.

There are certain special cases that may arise in the determination of a sequence characterizing H_n.

If the field H_r is exteriorly normal over all of the boundary of C_r, C_{r-1} must necessarily be chosen as the complete boundary of C_r, and hence be itself without boundary. If C_{r-1} be for any reason without boundary, C_{r-2} must necessarily be chosen as a null complex. Or again if H_r is never exteriorly normal to the boundary of C_r, C_{r-1} may be chosen as a null complex. If for any reason C_r is chosen as a null complex we take all of the complexes

$$C_r \cdots C_0$$

as null complexes, and regard the corresponding fields as empty.

A sequence characterizing H_n always exists when the singularities of each of the fields H_r are isolated. We shall prove in § 8 that a sequence characterizing H_n always exists provided that C_n and H_n be given analytically, regardless of whether the singularities of the fields H_r are isolated or not.

We shall now prove a very general theorem in which the fields on the sequence of complexes characterizing H_n will no longer be assumed to have isolated singularities, the field H_n for the present alone excepted.

THEOREM 3. *If $C_0C_1 \cdots C_n$ be a sequence of complexes characterizing H_n, and if the singularities of H_n are isolated, then*

$$I_n = E_0 - E_1 + \cdots + (-1)^n E_n, \qquad (n > 0)$$

where I_n is the index-sum of H_n, and *E_i is the characteristic of C_i.*

The method of proof will be that of mathematical induction.

If one recalls that E_0 is the number of points at which the field H_1 is directed toward the exterior of C_1 on the boundary of C_1, it readily appears that the theorem holds for $n = 1$.

Consider now the field H_{n-1} on C_{n-1}. In the special case where C_{n-1} is null Theorem 3 reduces to Theorem 2. If, on the other hand, C_{n-1} is not null, refer Σ_n in the neighborhood of any point of B_{n-1} to geodesic coördinates (x), as in § 5. Let C_{n-1} be regularly subdivided so finely that the set $\bar{C}_{n-1}$ of $(n-1)$-cells of C_{n-1} that contain as interior or boundary points, points at which H_{n-1} has singularities will contain only points at which the n-th component h_n of H_n is positive. According to Lemma A § 4, the field H_{n-1} on B_{n-1} can be continuously deformed without change of H_{n-1} except on $\bar{C}_{n-1}$, into a new field which has within $\bar{C}_{n-1}$ at most a finite set π of singularities. In this deformation let t be the parameter, and vary from 0 to 1 inclusive. At the point $(x_1, \cdots, x_{n-1}, 0)$ of B_{n-1} let

$$\phi_i(x_1, \cdots, x_{n-1}, t) \qquad (i = 1, 2, \cdots, n-1)$$

be the components of the field at each instant t during the deformation.

We now return to H_n. Let e be an arbitrarily small positive constant. We shall extend the definition of H_n so as to include the points (x) for which

$$0 < x_n \leqq e \tag{1}$$

To define H_n for the points (1) we set

$$h_i(x_1, \cdots, x_n) = \phi_i(x_1, \cdots x_{n-1}, x_n/e) \qquad (i = 1, 2, \cdots, n-1) \tag{2}$$

$$h_n(x_1, \cdots, x_n) = h_n[x_1, \cdots, x_{n-1}, 0]. \tag{3}$$

Denote the new boundary, $x_n = e$, by B'_{n-1}. The geodesics normal to B_{n-1} establish a homeomorphism F between B_{n-1} and B'_{n-1} in which points on the same geodesic correspond. Under F let C_j on B_{n-1} correspond to C_j' on B'_{n-1}. $j = 0, 1, 2, \cdots, n-1$.

Because h_n is positive on $\bar{C}_{n-1}$, and because of (3), the extension of H_n has been one in which no new singularities have been introduced. The points on B'_{n-1} at which H_n is exteriorly normal to B'_{n-1} are simply the images under F of the points of the set π.

Let H'_{n-1} be the projection of H_n on B'_{n-1}. The complexes

$$C'_{n-1}C'_{n-2}\cdots C_0', \tag{4}$$

and the field H'_{n-1} satisfy the hypotheses of Theorem 3 for the case where n is replaced by n-1. That is, the singularities of H'_{n-1} are isolated, for they occur only at the points of π, and the complexes (4) form a characterizing sequence for H'_{n-1}. According to the principle of mathematical induction we may use Theorem 3 to infer that

$$I'_{n-1} = E_0 - E_1 + \cdots + (-1)^{n-1}E_{n-1}, \tag{5}$$

where I'_{n-1} is the index-sum of H'_{n-1} on C'_{n-1}, and where E_i is the characteristic of C_i and hence of C_i's image C'_i.

But since H_n is exteriorly normal to B_{n-1} at most at isolated points, Theorem A_o applies and gives

$$I_n = (-1)^n E_n + I'_{n-1}\ . \tag{6}$$

Theorem 3 now follows from (5) and (6).

If there be given any sequence of complexes

$$C_0C_1\cdots C_n \tag{7}$$

it will be convenient to term the corresponding sum

$$E_0 - E_1 + \cdots + (-1)^n E_n \tag{8}$$

the alternating characteristic of the sequence (7).

7. *The generalized index of infinite sets of singularities.* The index of an isolated singularity as defined in § 4 will be here called an *elementary index.* The extension of the notion of index to infinite sets of singularities of H_n will be in two directions.

First consider any set s of singularities of H_n which are all interior to some regular element S_n with regular coördinates (x). In the space of the coördinates (x) suppose the points of s are all interior to a Jordan manifold

J_{n-1} which, together with its interior D_n, lies on S_n. Let R_{n-1} be a unit $(n-1)$-sphere in the space (x). We orient R_{n-1} and J_{n-1} similarly. The *generalized index* v of the singularities of s will be taken as the degree of that transformation F of J_{n-1} onto R_{n-1} which is defined by H_n on J_{n-1}. The proper use of this index requires a proof of the following.

The index v is independent of the choice of J_{n-1} among Jordan manifolds chosen as above.

For suppose J'_{n-1} and D'_n were, respectively, a second choice of a Jordan manifold and its interior domain. Let C_n be regularly subdivided so finely that the complex $\bar{C}_n$ which is made up of all n-cells that contain points of s on their interiors or boundaries, lies wholly interior to both D_n and D'_n. According to Lemma A, H_n can be replaced by a new field identical with H_n at the points exterior to $\bar{C}_n$ but having a finite set of singularities s' within $\bar{C}_n$.

In D_n one can chose a set C'_n of n-cells of a sufficiently fine regular division of C_n in such a manner that C'_n contains the points of s' as interior points and is connected with respect to n-cells. According to Lemma B the new field can now be replaced by a field K_n identical with the previous one except within C'_n where K_n will have just one singularity P, and that with an elementary index equal to the elementary index-sum, say v', of s'.

But in all these changes of the given field we have not altered the field on J_{n-1} so that the degree of the transformation F of J_{n-1} on R_{n-1} is the same for the original field as for the altered field. But for the altered field with its isolated singularity P this degree, by the very definition of an elementary index, is the elementary index of P, and must then equal v'. Thus the generalized index of s as determined from J_{n-1} equals v'. Similarly the generalized index of s as determined from J'_{n-1} will also equal v'. Thus the generalized index determined from J_{n-1} equals that determined from J'_{n-1}, and the proof is complete.

We can now extend the definition of the index beyond that of sets of singularities contained in a regular element. We suppose simply that the set s is interior to a regular sub-complex $\bar{C}_n$ of C_n upon which there are no other singularities than those of s, and that the field H_n on $\bar{C}_n$ possesses a characterizing sequence of complexes. *This sequence of complexes will also be said to characterize s.*

We now define the index of s to be the alternating characteristic of any sequence of complexes characterizing s.

Practically this index is easy to determine, for example, for the case of a closed curve or surface of singularities in an analytic problem. One can in

general determine the index of such sets of singularities if one knows the nature of nearby vectors. In the case of a closed curve in space, for example, one can enclose the curve in a torus-like surface, and by considering the vectors on this torus-like surface obtain a characterizing sequence, and hence an index. Theoretically one needs the justification of the following lemma.

The generalized index of a set of singularities is independent of the particular sequence of complexes used to characterize it.

Suppose C_j' and C_j'' $(j = 0, 1, \cdots, n)$ are two sequences of complexes characterizing fields H_n' and H_n'', both of which contain a set of singularities s, and no others.

As in the proof of the preceding lemma we can here replace the given vector field by a new one which does not differ from the old either on the boundary of C_n' or of C_n'', but which has on C_n' and C_n'' a finite set of singularities s^0. By virtue of Theorem 3 the alternating characteristics of our two characterizing sequences will both equal the elementary index-sum I_n of s^0, and thus be equal to each other. Thus the statement in italics is proved.

In case an index can be assigned to a set of singularities in more than one of the preceding ways, it is readily seen by means of the methods of the preceding proofs that the resulting index, or index-sum will always be the same.

If there be given on C_n any field which possesses a characterizing sequence, its singularities can always be grouped into a finite number of distinct sets each of which has a generalized index. The sum of these indices will be called the *generalized index-sum*. Our final theorem is the following.

THEOREM 4. *If there be given on C_n a vector field H_n which possesses a characterizing sequence, the generalized index-sum of the field equals the alternating characteristic of the characterizing sequence.*

To prove the theorem we consider separately each set s to which an index v has been assigned, and replace the given field by a new field which differs from the old only in the neighborhood of the points of s, and which in that neighborhood has at most isolated singularities. The latter will have an elementary index-sum equal to v. The application of Theorem 3 to the field thereby resulting will give the present theorem.

Corollary. If the alternating characteristic is not zero the field must have at least one singular point.

8. ***Proof of the existence of sequences characterizing an analytic field.*** We shall now prove the following.

If C_n and H_n are analytic a particular choice of a sequence of complexes characterizing H_n may always be made.

The cosine of the angle between H_n and the exterior normal to the boundary B_{n-1} of C_n will be an analytic function f of any regular analytic parameters of B_{n-1}. The domain on B_{n-1} in which f is positive will, in general, be bounded, if at all, by analytic manifolds without singularities, and in that case we can choose C_{n-1} of the characterizing sequence as the domain $f \geqq 0$. To proceed consider the projection H_{n-1} of H_n on C_{n-1}. Using H_{n-1} and C_{n-1} as we have just used H_n and C_n we arrive at a definition of C_{n-2}, and so on.

In the special case where the locus $f = 0$ is not free from singularities, the locus $f = e$ will be free from singularities if e be a sufficiently small positive constant. We distinguish between the cases $f \equiv 0$ and $f \not\equiv 0$.

If $f \equiv 0$ the vectors of H_n will all be tangent to B_{n-1}, and C_{n-1} may properly be chosen as a null complex. This is consistent with the fact that $f \geqq e$ is here never satisfied.

Suppose $f \not\equiv 0$. The points at which the locus $f = 0$ is singular satisfy

$$f_{x_1} = f_{x_2} = \cdots = f_{x_n} = 0. \tag{1}$$

We regard $(x) = (x_1, \cdots x_n)$ for the moment as complex variables, with values neighboring real values. The points at which (1) is satisfied in the neighborhood of a real initial solution (x^o) of (1) will be made up, according to Weierstrass's second law* of the implicit fnuction theory, either of the point (x^o) alone, or of one or more connected "Gebilde" g. Each g has the property that any two points on it can be joined by a regular curve h along which the solution (x) of (1) is an analytic function of the independent variables of g except possibly at the end points of h. If (x^o) satisfies $f = 0$ we can infer from (1) that $f \equiv 0$ on each g.

The real points at which $f = e$ can never satisfy (1) if e be sufficiently small. For if such were the case one could use the Weierstrass-Bolzano cluster point theorem to prove the existence of a point on B_{n-1} at which f would be zero, and at which (1) would be satisfied, and in each of whose neighborhoods there would be points at which (1) would be satisfied and $f \neq 0$, contrary to the nature of the domains g just described.

Thus for a sufficiently small positive constant e, $f = e$ will be free from singularities, and $f \geqq e$ will be an admissible choice of C_{n-1}. We proceed similarly to the choices of C_{n-2}, C_{n-3}, etc.

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* Osgood, *Lehrbuch der Funktionentheorie II'*.

9. *Examples.* Examples of the simplest sort will now be given to illustrate the application of Theorem 4. Each field is supposed null only at interior points of its domain.

Example 1. Let there be given a field interior to a circle, with the field directed exteriorly on one boundary semicircle C_1, but directed interiorly on the other. Suppose the projection of the field on the boundary is directed exteriorly to C_1 at C_1's end points. Then the index sum I_2 of the field is given as follows:

$$I_2 = E_0 - E_1 + E_2 = 2 - 1 + 1 = 2.$$

Example 2. Let the field be given within a sphere, and on the sphere be interiorly directed except within a ring-shaped region. Let the projection of the field on the sphere be directed into the ring on the boundary of the ring. Then

$$I_2 = E_0 - E_1 + E_2 - E_3 = 0 - 0 + 0 - 1 = -1.$$

Examples of greater complexity in higher dimensions can equally well be given.

The application of the results of this paper to the theory of fixed points of deformations, or to coincidences of transformations of the same class are sufficiently obvious to require no further elucidation.

A Complete Solution of LaPlace's Equation by an Infinite Hypervariable.

By P. W. KETCHUM.

1. *Introduction:* In a former paper * the writer has shown that most of the important theorems pertaining to analytic functions of a complex variable also hold for analytic functions of hypervariables provided the commutative and associative laws are satisfied. Also, while there is no variable of three units over the field of real numbers which is harmonic, i. e., such that every analytic function satisfies Laplace's equation, there are many restricted variables of more than three units which have that property. But in all cases the functions of hypervariables possess line singularities so that no hypervariable furnishes the totality of harmonic functions.

In that paper, however, the hypervariables were assumed to have a *finite* number of units. In the present paper a restricted hypervariable will be studied which possesses an infinity of units, and whose analytic functions form the totality of harmonic functions.

$$
\begin{array}{ccc|c|ccc}
\vdots & & & & & & \\
\frac{1}{2}e_{-6} & \frac{1}{2}(-e_{-1}+e_{-5}) & \frac{1}{2}(e_{-2}+e_{-4}) & e_3 & \frac{1}{2}(-e_2+e_4) & \frac{1}{2}(e_1+e_5) & \frac{1}{2}(-e_0+e_6) \\
\frac{1}{2}(e_{-1}+e_{-5}) & \frac{1}{2}e_{-4} & \frac{1}{2}(e_{-1}+e_{-3}) & e_2 & \frac{1}{2}(-e_1+e_3) & \frac{1}{2}(e_0+e_4) & \frac{1}{2}(e_1+e_5) \\
\frac{1}{2}(-e_{-2}+e_{-4}) & \frac{1}{2}(-e_{-1}+e_{-3}) & \frac{1}{2}e_{-2} & e_1 & \frac{1}{2}(-e_0+e_2) & \frac{1}{2}(-e_1+e_3) & \frac{1}{2}(-e_2+e_4) \\
\hline
\cdots\ e_{-3} & e_{-2} & e_{-1} & e_0 & e_1 & e_2 & e_3\ \cdots \\
\hline
\frac{1}{2}(-e_2-e_4) & \frac{1}{2}(-e_1-e_3) & \frac{1}{2}(-e_0-e_2) & e_{-1} & \frac{1}{2}e_{-2} & \frac{1}{2}(e_{-1}+e_{-3}) & \frac{1}{2}(e_{-2}+e_{-4}) \\
\frac{1}{2}(e_1-e_5) & \frac{1}{2}(e_0-e_4) & \frac{1}{2}(-e_1-e_3) & e_{-2} & \frac{1}{2}(-e_{-1}+e_{-3}) & \frac{1}{2}e_{-4} & \frac{1}{2}(-e_{-1}+e_{-5}) \\
\frac{1}{2}(-e_0-e_6) & \frac{1}{2}(e_1-e_5) & \frac{1}{2}(-e_2-e_4) & e_{-3} & \frac{1}{2}(-e_{-2}+e_{-4}) & \frac{1}{2}(e_{-1}+e_{-5}) & \frac{1}{2}e_{-6} \\
 & & & \vdots & & &
\end{array}
$$

* "Analytic Functions of Hypercomplex Variables," *Transactions of the American Mathematical Society*, Vol. 30 (1928) pp. 641-667. References and the meaning of the more common symbols are found in that paper. In Eq. 1 of that paper, the second semicolon should be a subscript j. On pp. 660 and 661, the equation $w_1 = x_1$ should read $w_i = x_i$. The ϵ_k's in (4) are defined by the relation $\epsilon = \sum \epsilon_k e_k$.

2. *Definition of the Variable:* The variable to be studied is

$$w = ze_{-1} + xe_0 + ye_1, \tag{1}$$

where e_{-1}, e_0 and e_1 are three of an infinite set of units whose laws of multiplication are given in the accompanying table, (e. g., $e_{-1}e_2 = \frac{1}{2}(e_{-1} + e_{-3})$, etc.), and e_0 is the modulus of the algebra. The table shows that, disregarding the signs, the subscripts of the units of the product of two units are the sum and difference of the subscripts of the factors.

A direct computation shows that the products in the table are obtained if we put *

$$e_n = i^n \cos n\alpha, \qquad e_{-n} = i^n \sin n\alpha, \qquad [i = (-1)^{\frac{1}{2}}, \quad n = 1, 2, \cdots], \tag{2}$$

where the functions on the right can be combined functionally in any way, but are not to be regarded as a collection of function values (that is, $\cos(m - n)\alpha$ cannot be identified with $\cos(n - m)\alpha$, etc.).

Let $a = \sum a_i e_i$ and $b = \sum b_i e_i$, be any two numbers of the algebra and let $c = \sum c_i e_i$ be the product ab. Then,

$$c_0 = \tfrac{1}{2}a_0b_0 + \tfrac{1}{2}\sum(-1)^n a_n b_n.$$

From this relation it is evident that values of a and b exist for which c_0 is infinite, so that *the product of two numbers may not exist.*

If $\sum |a_i|$ *converges, then a is said to be absolutely convergent.* Consider two absolutely convergent numbers a and b. If $c = ab$, all the terms of c_k will be of the form $\gamma_{ijk}a_ib_j$ where the γ's are 0, 1, or $\pm\frac{1}{2}$. If each such term is replaced by its absolute value, the sum, d_k, will be greater than or equal to $|c_k|$ (if d_k and $|c_k|$ exist). Suppose the terms of the series $\sum d_k$ are formally rearranged so as to form the double series $\sum \lambda_{ij} |a_ib_j|$, where $\lambda_{ij} = \sum |\gamma_{ijk}|$. Then for i and j not zero and $i \neq -j$, $\lambda_{ij} = 1$; if either i or j vanishes, $\lambda_{ij} = 1$; and if $i = -j \neq 0$, $\lambda_{ij} = \frac{1}{2}$. Hence this double series is term by term less than or equal to the absolutely convergent series $\sum |a_ib_j|$, and must therefore be itself absolutely convergent. A valid rearrangement of the terms may now be carried out so as to change the double series back to the form $\sum d_k$, showing that this series converges absolutely. But this series dominates the series $\sum |c_k|$ which must as a result also converge absolutely, thus showing that the number c is absolutely convergent. Hence, *the product of two absolutely convergent numbers is absolutely convergent.*

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* The writer is much indebted to Prof. J. B. Shaw for these remarkable relations.

Any number having components only in a finite number of directions is absolutely convergent, so the product of any such number with any absolutely convergent number is absolutely convergent. Hence w^n *is absolutely convergent* for n any positive integer and for all values of w. Also, *if either a or b has components in only a finite number of directions, the product ab always exists,* since the expression for any of the components of ab will contain only a finite number of terms.

The multiplication table shows immediately that for any two numbers a and b, $ab = ba$ provided one of the products is absolutely convergent. It is also obvious when the units are expressed as in (2) that for any three numbers, a b and c, $(ab)c = a(bc)$ provided one member is absolutely convergent.

If division is defined in the ordinary manner *the quotient of two numbers is not unique.*

In the following, a three dimensional region or volume will be represented by τ or V, a two dimensional region or surface by S, and a one dimensional region or curve by C. Only curves and surfaces which are sufficiently smooth will be considered, and only those volumes which are bounded by such surfaces. Also, all volumes and surfaces will be assumed to be simply connected. Any multiply connected regions can be made simply connected by the introduction of cuts in the usual manner. The following discussion is limited to single valued functions, but most of the results will hold for multiple valued functions by first making the function single valued by the introduction of Riemann spaces in a manner analogous to the treatment of multiple valued functions in complex variables. For simplicity, all real (conjugate) functions will be taken as analytic, although most of the results undoubtedly could be stated for more general classes of functions.

3. *Differentiation:* Since division is not unique it might be expected that differentiation in the ordinary sense would fail, but such is not the case. If, for a function $f(w) = \sum f_i(x, y, z)$, we define the *derivative* (at a point w where the conjugate functions are analytic) by the relation $df = f'dw$, and require that f' *be independent of the value of dw,* there is obtained, by a method precisely the same as is used in obtaining the Sheffer's equations for a finite variable, the following system of equations:

$$\partial f_n/\partial x = f'_n; \qquad (n = -\infty, \cdots, \infty) \tag{3A}$$

$$\begin{cases} \partial f_{\pm n}/\partial y = \tfrac{1}{2}(f'_{\pm n\mp 1} - f'_{\pm n\pm 1}); & \partial f_0/\partial y = -\tfrac{1}{2}f'_1; \\ \qquad (n = 2, \cdots, \infty) & \\ \partial f_1/\partial y = f'_0 - \tfrac{1}{2}f'_2; & \partial f_{-1}/\partial y = -\tfrac{1}{2}f'_{-2}; \end{cases} \tag{3B}$$

$$(3C)\quad \begin{cases} \partial f_{\pm n}/\partial z = \pm \tfrac{1}{2}\,(f'_{\mp n\pm 1} + f'_{\mp n\mp 1})\,; & \partial f_0/\partial z = -\tfrac{1}{2} f'_{-1}\,; \\ \qquad\qquad (n = 2, \cdots, \infty) & \\ \partial f_1/\partial z = -\tfrac{1}{2} f'_{-2}\,; & \partial f_{-1}/\partial z = f'_0 + \tfrac{1}{2} f'_2. \end{cases}$$

These are necessary and sufficient conditions that f' be independent of dw.

The functions f'_n can be eliminated from these equations in several ways, giving equations analogous to the Cauchy-Riemann or Sheffer's equations. The simplest method of elimination is the substitution of (3A) in (3B) and (3C), giving

$$(4A)\quad \begin{cases} \partial f_{\pm n}/\partial y = \tfrac{1}{2}[(\partial f_{\pm n\mp 1}/\partial x) - (\partial f_{\pm n\pm 1}/\partial x)]\,; & \partial f_0/\partial y = -\tfrac{1}{2}(\partial f_1/\partial x); \\ \qquad\qquad (n = 2, \cdots, \infty) & \\ \partial f_1/\partial y = \partial f_0/\partial x - \tfrac{1}{2}\partial f_2/\partial x\,; & \partial f_{-1}/\partial y = -\tfrac{1}{2}\partial f_{-2}/\partial x\,; \end{cases}$$

$$(4B)\quad \begin{cases} \partial f_{\pm n}/\partial z = \mp \tfrac{1}{2}[(\partial f_{\mp n\pm 1}/\partial x) + (\partial f_{\mp n\mp 1}/\partial x)]\,; & \partial f_0/\partial z = -\tfrac{1}{2}(\partial f_{-1}/\partial x); \\ \qquad\qquad (n = 2, \cdots, \infty) & \\ \partial f_1/\partial z = -\tfrac{1}{2}(\partial f_{-2}/\partial x)\,; & \partial f_{-1}/\partial z = (\partial f_0/\partial x) + \tfrac{1}{2}(\partial f_2/\partial x). \end{cases}$$

A function will be said to be *monogenic* if it possesses a derivative for all values of w in some region τ and if its conjugate functions are analytic in τ. In order for a function to be monogenic, it is necessary that its conjugate functions satisfy (4). These conditions, together with the analyticity of the conjugate functions, are also sufficient, since, if we put $f'_n = \partial f_n/\partial x$, (4) reduce to (3). Hence, *for any given set of real analytic functions $f_i(x, y, z)$, in order for the function $f(w) = \sum f_i$ to be monogenic, it is necessary and sufficient that the functions f_i satisfy Eqs.* 4 *for all values of x, y, and z, in some region τ.*

From (4), the following *necessary conditions for monogenicity* are obtained, by elimination of the x derivatives:

$$(5)\quad \begin{aligned} &\partial f_{-1}/\partial y = \partial f_1/\partial z\,; && \tfrac{1}{2}(\partial f_{-2}/\partial y) = \tfrac{1}{2}(\partial f_2/\partial z) - (\partial f_0/\partial z)\,; \\ &(\partial f_0/\partial y) + \tfrac{1}{2}(\partial f_2/\partial y) = -\tfrac{1}{2}(\partial f_{-2}/\partial z)\,; && \\ &(\partial f_{\mp n\pm 1}/\partial y) + (\partial f_{\mp n\mp 1}/\partial y) = (\partial f_{\pm n+1}/\partial z) - (\partial f_{\pm n-1}/\partial z)\,; && (n = 2, \cdots, \infty). \end{aligned}$$

and also the equations:

$$(6)\quad \begin{aligned} &(\partial f_0/\partial x) - \tfrac{1}{2}(\partial f_1/\partial y) - \tfrac{1}{2}(\partial f_{-1}/\partial z) = 0\,; \\ &(\partial f_1/\partial x) + (\partial f_0/\partial y) - \tfrac{1}{2}(\partial f_2/\partial y) - \tfrac{1}{2}(\partial f_{-2}/\partial z) = 0\,; \\ &(\partial f_{-1}/\partial x) - \tfrac{1}{2}(\partial f_{-2}/\partial y) + (\partial f_0/\partial z) + \tfrac{1}{2}(\partial f_2/\partial z) = 0\,; \\ &(\partial f_{\pm n}/\partial x) + \tfrac{1}{2}(\partial f_{\pm n\mp 1}/\partial y) - \tfrac{1}{2}(\partial f_{\pm n\pm 1}/\partial y) \mp \tfrac{1}{2}(\partial f_{\mp n\pm 1}/\partial z) \\ &\qquad\qquad \mp (\partial f_{\mp n\mp 1}/\partial z) = 0. \qquad (n = 2, \cdots, \infty). \end{aligned}$$

If the derivatives $\partial^2 f_n/\partial x^2$, $\partial^2 f_n/\partial y^2$ and $\partial^2 f_n/\partial z^2$ are formed from (4) and (6) and summed, it is found that $\Delta f_n = 0$ for every n. Hence, *all the conjugate functions of any monogenic function of w are harmonic.*

By differentiating (3) and eliminating the derivatives $\partial^2 f_n/\partial x \partial y$, etc. we obtain (4) expressed in terms of f'_n instead of f_n. Hence, *the derivatives of a monogenic function are themselves monogenic in the same region.*

It is obvious that the function w is monogenic for all values of w. By direct computation, it can easily be shown that $d(wf) = wdf + fdw$. Now if $f(w)$ is monogenic in a region τ, $df = f'dw$. Substituting this value of df in the above expression for $d(wf)$ we see that wf is monogenic in τ and has the derivative $wf' + f$. Hence, *w^n is monogenic for all values of w and for n any positive integer, and has the derivative nw^{n-1}.*

4. *Integration:* The *line integral* of a function over a curve C is defined as follows:

$$
\begin{aligned}
\int_C f(w)dw = {} & e_0 \int_C f_0 dx - \tfrac{1}{2} f_1 dy - \tfrac{1}{2} f_{-1} dz \\
& + e_1 \int_C f_1 dx + (f_0 - \tfrac{1}{2} f_2)\, dy - \tfrac{1}{2} f_{-2} dz \\
& + e_{-1} \int_C f_{-1} dx - \tfrac{1}{2} f_{-2} dy + (f_0 + \tfrac{1}{2} f_2)\, dz \\
& + \sum_{n=2}^{\infty} e_{\pm n} \int_C f_{\pm n} dx + \tfrac{1}{2}(f_{\pm n \mp 1} - f_{\pm n \pm 1})\, dy \mp \tfrac{1}{2}(f_{\pm n \mp 1} + f_{\pm n \mp 1})\, dz.
\end{aligned} \tag{7}
$$

Let $f(w)$ be any monogenic function and C a closed curve corresponding to which there can be found a diaphragm S, whose complete boundary is C, and every point of which is a regular point of $f(w)$. Then

$$\int_C f(w)dw = 0. \tag{8}$$

To prove this statement, the integrals of Eq. 7 are converted into surface integrals over S by Stokes theorem, thereby giving integrands, each of which vanishes at every point of S by virtue of (4) and (5).

The converse of the above theorem is also true, namely, *if a given set of real functions $f_i(x, y, z)$ are analytic in a region τ and if for $f(w) = \sum f_i$, $\int_C f(w)dw = 0$ for every closed curve C in τ, then $f(w)$ is monogenic in τ.*

If $f(w)$ is monogenic in a region τ, and if $F(w) = \int_a^w f(w)dw$, it is

evident from Eq. 7 that $F(w)$ has a derivative equal to $f(w)$. Hence, since the conjugate functions of $F(w)$ must be analytic, $F(w)$ is monogenic in τ.

If $\phi(w)$ is any function such that $\phi'(w) = f(w)$, then $F(w)$ and $\phi(w)$ can differ by at most a constant, and there follows at once *the fundamental theorem of integral calculus:*

$$\int_\alpha^\beta f(w)\,dw = \phi(\beta) - \phi(\alpha). \tag{9}$$

The *surface differential* $d\omega$ will be defined as follows:

$$d\omega = dydze_0 + dxdze_1 + dxdye_{-1}, \tag{10}$$

and *surface integration* will be defined as follows:

$$\begin{aligned}\int_S f(w)\,d\omega = e_0 \int\int_S & f_0 dydz - \tfrac{1}{2}f_1 dxdz - \tfrac{1}{2}f_{-1}dxdy \\ + e_1 \int\int_S & f_1 dydz + (f_0 - \tfrac{1}{2}f_2)\,dxdz - \tfrac{1}{2}f_{-2}dxdy \\ + e_{-1} \int\int_S & f_{-1}dydz - \tfrac{1}{2}f_{-2}dxdz + (f_0 + \tfrac{1}{2}f_2)\,dxdy \\ + \sum_{n=2}^{\infty} e_{\pm n} \int\int_S & f_{\pm n}dydz + \tfrac{1}{2}(f_{\pm n\mp 1} - f_{\pm n\pm 1})\,dxdz \\ & \mp \tfrac{1}{2}(f_{\pm n\pm 1} + f_{\pm n\mp 1})\,dxdy.\end{aligned} \tag{11}$$

Now suppose that the surface S is closed, and the function $f(w)$ is monogenic at every point within and on S. The integrals of Eq. 11 can be converted by Green's theorem into volume integrals taken over the space within S. But when this is done, it is found that in each volume integral the integrand vanishes by virtue of (6). *Hence;*

$$\int_S f(w)\,d\omega = 0 \tag{12}$$

in a region τ throughout which $f(w)$ is monogenic.

If a surface integration is carried out from a fixed loop (or point) L_1, to a variable loop L, the integral $\int_{L_1}^{L} f(w)\,d\omega$ is independent of the surface between the two loops, as long as the surface lies entirely within τ. Hence the integral defines what may be called a *loop function* $F(L)$, i. e. for every loop in the region of monogenicity there corresponds a hypernumber.

5. *Completeness of Solution of LaPlace's Equation:* Let $f_0(x, y, z)$ be any harmonic function which is analytic in a region τ. Integrating equa-

tions 4A and the second equation of 4B, there is obtained step by step a set of functions f_i all defined in terms of f_0, as follows:

$$f_1 = -2\int_a^x (\partial f_0/\partial y)\,dx + \phi_1(y,z); \qquad f_{-1} = -2\int_a^x (\partial f_0/\partial z)\,dx + \phi_{-1};$$

$$f_2 = 2f_0 - 2\int_a^x (\partial f_1/\partial y)\,dx + \phi_2; \qquad f_{-2} = -2\int_a^x (\partial f_{-1}/\partial y)\,dx + \phi_{-2};$$

(13)

$$f_{\pm n} = f_{\pm n\mp 2} - 2\int_a^x (\partial f_{\pm n\mp 1}/\partial y)\,dx + \phi_{\pm n}; \qquad (n = 3 \cdots \infty),$$

where α is any point in τ and x any point in τ which can be reached from α by varying x only, and not passing outside of τ. If from the functions defined in (13), there is formed the function $f(w) = \sum f_n e_n$, in order for $f(w)$ to be a monogenic function it is sufficient that the functions f_i satisfy Eqs. 4 in a subregion of τ. By their definitions it is obvious that they must satisfy Eqs. 4A. Substitution shows that they also satisfy Eqs. 4B if, and only if, the functions ϕ_i satisfy the following conditions:

(14)

$$\begin{aligned}
&\partial\phi_1/\partial z = \partial\phi_{-1}/\partial y, && (\partial\phi_1/\partial y) + (\partial\phi_{-1}/\partial z) = 2(\partial f_0/\partial x)_\alpha;\\
&\partial\phi_2/\partial z = \partial\phi_{-2}/\partial y, && (\partial\phi_2/\partial y) + (\partial\phi_{-2}/\partial z) = 2(\partial f_1/\partial x)_\alpha;\\
&(\partial\phi_{\pm n}/\partial y) \pm (\partial\phi_{\mp n}/\partial z) = -2(\partial f_{\pm n\mp 2}/\partial y)_\alpha && (n = 3 \cdots \infty).
\end{aligned}$$

In this set of equations, for every pair of functions ϕ_n and ϕ_{-n} there is a pair of equations. Proceeding step by step it follows from the existence theorem for solutions of simultaneous first order partial differential equations * that there always exist functions ϕ_n which are analytic and satisfy (14) in τ. Functions f_n defined by Eqs. 13, therefore, always exist which satisfy Eqs. 4 and therefore form a function which is monogenic in some subregion of τ. It follows that *the totality of conjugate functions of monogenic functions of w will include all harmonic functions which are analytic.* Hence, we have obtained a complete solution of Laplace's equation in three variables in the same sense that the analytic functions of a complex variable form a complete solution of Laplace's equation in two variables.

6. *Expansion in series:* If

$$\operatorname*{Lim}_{w\to\alpha} f_i(x, y, z) = A_i,$$

for every value of i, then $f(w)$ will be said to have the limit A as w approaches α. A function $f(w)$ will be said to be continuous at a point α if its conjugate

* Goursat-Hedrick, *Mathematical Analysis*, Vol. 2, Pt. 2, p. 57.

functions are continuous at that point. A monogenic function is evidently continuous in its region of monogenicity. A series will be said to be convergent, absolutely convergent, or uniformly convergent if its conjugate series are all convergent, absolutely convergent or uniformly convergent, respectively. Also, the sum of a convergent series is defined by the equation,

$$f(w) = \sum_{j=0}^{\infty} U_j(w) = \sum_{i=-\infty}^{\infty} \sum_{j=0}^{\infty} U_{ji}(x, y, z) e_i.$$

From these definitions the following theorems are easily obtained:

1. A series which is uniformly convergent in a region and whose terms are analytic in τ defines a function which is continuous in τ.

2. If $f(w) = \sum_{0}^{\infty} U_j(w)$, and if this series is uniformly convergent along a curve C, and if the terms are analytic on C, then

$$\int_C f(w)\,dw = \sum_{0}^{\infty} \int_C U_j(w)\,dw.$$

3. If, in a formal manner, the series $\sum U'_j(w)$ is formed, and if this derived series converges uniformly in a region τ, then for values of w in τ,

$$f'(w) = \sum_{0}^{\infty} U'_j(w).$$

4. A series which converges uniformly in a region τ, and whose terms are analytic in τ, defines a function which is monogenic in τ. This theorem can be proved with the aid of the analogue of Morera's theorem.

As a corollary to this theorem it follows that if a power series in $w - \alpha$ converges at every point of a region τ, which includes α in its interior, then $f(w)$ is monogenic in τ.

A function will be said to be *analytic* in a region τ if it can be expanded in a power series about any point in the interior of τ. A function will be called an analytic function if it is analytic in some region τ. From the above corollary it follows that *every function which is analytic in a region τ is monogenic within τ*; and every analytic function is also a monogenic function.

It can also be shown by a method entirely analogous to that used in finite hypervariables (l. c. Sec. 7) that if a function is monogenic in the neighborhood of a point α, it can be expanded in a power series in $w - \alpha$ which con-

verges in some region about α. Hence, *every function which is monogenic within a region is also analytic within that region*; and every monogenic function is also an analytic function.

From the last theorem there follows at once by the method of real variables the *Taylor expansion* of a function which is monogenic in the neighborhood of α, namely,

$$f(w) = \sum_0^\infty (e_0/j!) f^{(j)}(\alpha)(w-\alpha)^j; \tag{15}$$

the series converging for some region about α.

7. *Connection with spherical harmonics:* Let $(w^m)_n$ be the nth conjugate function of the function w^m. It can be shown by induction that these conjugate functions, when expressed in spherical coördinates (ρ, θ, ϕ) satisfy the following equations:

$$\begin{aligned} (w^m)_0 &= \rho^m P_m(\cos\theta), \\ (w^m)_{\pm n} &= \rho^m \frac{2m!}{(m+n)!} \begin{matrix}\cos \\ \sin\end{matrix} n\phi \quad P_m{}^n(\cos\theta), \quad (n \neq 0), \end{aligned} \tag{16}$$

where the functions $P_m{}^n$ are the associated Legendre functions. The proof makes use of recurrence relations existing between the conjugate functions, which are easily obtained from the multiplication table; and the following recurrence relations between the Legendre functions and their derivatives:

$$\begin{aligned} 2mP_m{}^{(i)}(\mu) = {} & (\mu^2-1)P_{m-1}{}^{(i+1)}(\mu) + 2(m+i)\mu P_{m-1}{}^{(i)}(\mu) \\ & + (m+i)(m+i-1)\,P_{m-1}{}^{(i-1)}(\mu), \qquad (i \neq 0), \end{aligned}$$

where the superscripts indicate number of differentiations. This formula can be proved by applying Leibnitz' formula for the nth derivative of a product to Rodrigue's formula for Legendre functions.

Now consider a power series in w of the form

$$S(w) = \sum_0^\infty \alpha^{(m)} w^m.$$

The function $S_0(w)$, which may properly be called the real part of $S(w)$ is,

$$S_0(w) = \sum_0^\infty S_0{}^{(m)}(w),$$

where the functions $S_0^{(m)}(w)$ are homogeneous and of degree m in x, y and z. By direct computation,

$$S_0^{(m)}(w) = \tfrac{1}{2}\alpha_0^{(m)}(w^m)_0 + \tfrac{1}{2}\sum_{n=-m}^{m}(-1)^n\alpha_n^{(m)}(w^m)_n.$$

Substituting values from Eq. 16, it follows that $S_0^{(m)}(w)$ is equivalent to $\rho^m Y_m(\theta, \phi)$ where Y_m is the surface spherical harmonic of degree m.

Hence $$S_0(w) = \sum_0^\infty \rho^m Y_m(\theta, \phi).$$

It has been shown that any harmonic function which is analytic about the origin can be expressed as a particular function $S_0(w)$. Hence, we have an independent proof of the theorem that any function which is harmonic and analytic in the neighborhood of the origin may be expanded in a series of spherical harmonics of the form $\sum_0^\infty \rho^m Y_m(\theta, \phi)$.

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Functions with Assigned Initial Values.

By W. J. Trjitzinsky.

1. *Introduction.* By initial values of a function are meant the values of the function and of its successive derivatives at a point. The problem of constructing indefinitely differentiable functions of a real variable with assigned initial values at a point has been undertaken by de la Vallée Poussin * in connection with the study of quasi-analytic functions; he gives the representation in the form $\sum_{j=1}^{\infty} \alpha_j \cos m_j x$. I have extended his method, introducing the representation of the form $\sum_{j=1}^{\infty} \alpha_j f(m_j x)$ † where $f(x)$ is an even power series restricted in a certain way. In this paper three distinct methods are offered for the solution of the problem in question; the first method [§ 2] giving an extension of both of the mentioned representations, and the third [§ 4] having an interest of its own in the study of a type of essentially singular points. An asymptotic representation will be given in connection with the second method [§ 3].

2. *Representation in a Series of Power Series.* We introduce a definition.

Definition. $f(x)=\sum_{k=0}^{\infty} b_k x^k$ *is of class* (A) *if for a fixed* h, $p=1,2,3,\cdots$, *and all real values of* x.

$$|f(x)| \text{ and } |f^{(p)}(x)| < h, f(0) \neq 0 \text{ and } f^{(p)}(0) \neq 0.$$

Examples of functions of class (A) are $\cos x + \sin x$, e^{ix}. If $f_1(x)$ is a function of class (A), indefinitely many functions of that class can be constructed by taking

$$f(x)=\sum_{j=1}^{\infty} d_j f_1(p_j x), \tag{1}$$

where $\sum_{j=1}^{\infty} |d_j|$ converges, and the p_j are real numbers and $|p_j| \leq 1$ $(j=1,$

* C. de la Vallée Poussin, *The Rice Institute Pamphlet*, Vol. 12 (1925), No. 2, pp. 164-172.

† W. J. Trjitzinsky, "Representation of Functions which are determined by their initial Values," *Annals of Mathematics*, Vol. 29 (December, 1927), pp. 73-78.

$2, \cdots)$; this is easily verified by taking the nth derivative of (1) and observing that it is bounded for all values of x and that its modulus is less than a constant independent of n.

We shall now determine the constants d_j so that the sum function $F(x) = \sum_{k=1}^{\infty} \alpha_k f(m_k x)$ has an assigned set of initial values $F(0)$, $F^{(n)}(0)$, at $x = 0$. Let $f(x)$ be of class (A), and let m_k be positive numbers, not necessarily integers, such that $m_{k+1}/m_k > \lambda > 1$, $(k = 1, 2, \cdots)$; also $F(0)$, $F^{(n)}(0)$ will be taken so that $|F(0)/f(0)|, |F^{(n)}(0)/f^{(n)}(0)| < G$, $(n = 1, 2, \cdots)$.

Consider

$$F_n(x) = a_1 f(m_1 x) + a_2 f(m_2 x) + \cdots + a_n f(m_n x). \tag{2}$$

Determine $a_1, a_2, \cdots, a_n$ by the equations:

$$\begin{aligned} &F(0)/f(0) = a_1 + a_2 + \cdots + a_n, \\ &F^{(1)}(0)/f^{(1)}(0) = m_1 a_1 + m_2 a_2 + \cdots + m_n a_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &F^{(n-1)}(0)/f^{(n-1)}(0) = m_1^{n-1} a_1 + m_2^{n-1} a_2 + \cdots + m_n^{n-1} a_n. \end{aligned} \tag{3}$$

Solving this system by determinants we observe that a solution of equations of the form (3) has been already given by Vallée-Poussin *; we need only to replace $(-1)^j C_{2j}$, wherever they occur in his proof, by $F(0)/f(0)$, $F^{(j)}(0)/f^{(j)}(0)$, and m_j^2 by m_j; we have then

$$a_k = (1/2\pi i) \int_C \frac{\phi_n(z^2) \cdot \psi(z)\, dz}{(z^2 - m_k)\, \phi_n^{(1)}(m_k)}, \quad (k = 1, 2, \cdots, n), \tag{4}$$

where

$$\phi_n(z) = \prod_{j=1}^{n} [1 - (z/m_j)],$$

$$\psi(z) = \sum_{j=0}^{\infty} [F^{(j)}(0)/f^{(j)}(0)] \cdot 1/z^{2j+1}$$

and C is a circle having $z = 0$ for center and a radius $R \geq G$. $(F^{(0)}(0) = F(0), f^{(0)}(0) = f(0))$.

When $n \to \infty$, $\phi_n(z^2) \to \phi(z^2) = \prod_{j=1}^{\infty} [1 - (z^2/m_j)]$, which converges; hence, as $n \to \infty$,

$$\lim a_k = \alpha_k = (1/2\pi i) \int_C \frac{\phi(z^2)\psi(z)\, dz}{(z^2 - m_k)\phi^{(1)}(m_k)} \tag{5}$$

* De la Vallée Poussin, *loc. cit.*, pp. 166-167.

and exists. Further, we find that

$$|a_k| \text{ and } |\alpha_k| < h'\left(\frac{m_1 m_2 \cdots m_{k-1}}{m_k^{k-1}}\right), \tag{6}$$

where h' is a positive constant independent of k and n.*

The inequalities (6), together with the boundedness of $f(x)$ and of its successive derivatives for all real values of x insure absolute and uniform convergence of the sum function $\sum_{j=1}^{n} a_j f(m_j x)$ and of its successive derivatives for all values of n, and also that of the limiting series $\sum_{j=1}^{\infty} \alpha_j f(m_j x)$ and its successive derivatives. We are now enabled to formulate the following theorem:

THEOREM I. *Let m_j be a sequence of positive numbers such that for all values of n*

$$m_{n+1}/m_n > \lambda > 1.$$

Let $f(x)$ be a function of class (A) and let $F(0)$, $F^{(j)}(0)$ $(j = 1, 2, \cdots)$ be numbers such that $|F(0)/f(0)|$ and $|F^{(j)}(0)/f^{(j)}(0)| < G$, a positive number. Coefficients α_j can then be so determined that the series

$$F(x) = \alpha_1 f(m_1 x) + \alpha_2 f(m_2 x) + \cdots + \alpha_n f(m_n x) + \cdots \tag{7}$$

and all of its derivatives will converge, and the sum function will have $F(0)$ $F^{(1)}(0)$, $F^{(2)}(0), \cdots$ for initial values at $x = 0$. The coefficients α_j can be given in the form of complex integrals (5).

We shall state now that when

$$\lim_{n\to\infty}\left(\frac{m_1 m_2 \cdots m_n}{m_{n+1}}\right) = 0 \tag{8}$$

the above representation is unique; that is, having assigned a function $f(x)$ of class (A) it is impossible to have two distinct functions of the form $F(x) = \sum_{j=1}^{\infty} \alpha_j f(m_j x)$, with an assigned set of initial values $F(0)$, $F^{(j)}(0)$ $(j = 1, 2, \cdots)$, $|F(0)/f(0)|$, $|F^{(j)}(0)/f^{(j)}(0)| < G$, $|\alpha_n| < h'\left(\frac{m_1 \cdots m_{n-1}}{m_n}\right)$ (h' = constant). This can be proved by a method which is an immediate extension of the proof of a similar theorem concerning trigonometric series.†

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* De la Vallée Poussin, *loc. cit.*, pp. 168-169.

† De la Vallée Poussin, *loc. cit.*, pp. 170-171.

DEFINITION. $F(x)=\sum_{j=1}^{\infty}\alpha_j f(m_j x)$ *where* $f(x)$ *is a specified function of class* (A), $|\alpha_n|<h\left(\frac{m_1 m_2\cdots m_{n+1}}{m_n{}^{n-1}}\right)$ *and* $\lim_{n\to\infty}\left(\frac{m_1 m_2\cdots m_n}{m_{n+1}}\right)=0$ *will be said to be of class* (B).

As the difference of two functions of class (B) is of class (B), it suffices to prove that, provided all the initial values are zero, a function of this class vanishes identically. The coefficients $a_1, a_2, \cdots, a_n$ must then satisfy a set of equations:

$$
\begin{aligned}
R_0 &= -(a_{n+1}+a_{n+2}+\cdots)=a_1+a_2+\cdots+a_n,\\
R_1 &= -(m_{n+1}\cdot a_{n+1}+m_{n+2}\cdot a_{n+2}+\cdots)\\
&\qquad = m_1a_1+m_2a_2+\cdots+m_na_n,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
R_{n-1} &= -(m^{n-1}{}_{n+1}\,a_{n+1}+m^{n-1}{}_{n+2}\,a_{n+2}+\cdots)\\
&\qquad = m_1{}^{n-1}a_1+m_2{}^{n-1}a_2+\cdots+m_n{}^{n-1}a_n.
\end{aligned}
\tag{9}
$$

We have, consequently

$$
a_k=(1/2\pi i)\int_0\frac{\phi_n(z^2)\cdot\sum_{j=1}^{\infty}R_j\cdot(1/z^{2j+1})}{(z^2-m_k)\,\phi_n{}^{(1)}(m_k)}dz,\qquad (k=1,2,\cdots,n). \tag{10}
$$

The first term in R_{n-1} [that is, $(-m^{n-1}{}_{n+1}\,a_{n+1})$] approaches 0, as $n\to\infty$, which follows from the inequalities satisfied by a_j, α_j; on the strength of these inequalities we have further that $\lim_{n\to\infty}R_{n-1}=0$; it is obvious then that each $R_j\to 0$, as $n\to\infty$. Hence, it follows from (10) that $\lim_{n\to\infty}a_k=0$ for all values of k. This demonstrates that a function of class (B) with zero initial values is identically zero.

We formulate the following theorem:

THEOREM II. *Functions of class* (B) *are determined uniquely by their initial values at* $x=0$.*

Let us return to Theorem I; the restriction on the initial values $F(0)$, $F^{(j)}(0)$ can be removed.† Suppose that $F(0)$, $F^{(j)}(0)$ is any set of constants; if $|F(0)/f(0)|$, $|F^{(j)}(0)/f^{(j)}(0)|$ are bounded, Theorem I can be applied; if not, then a function $G(x)$ can be constructed as follows: let $G(x)=F(0)$

---

* These functions are neither analytic nor quasi-analytic, similarly to the trigonometric case.

† Compare with the method used on p. 164 of the Pamphlet cited before.

$+\sum_{n=1}^{\infty} a_n f(m_n x)/m^n{}_n$. Differentiating in succession and letting $x=0$ we derive a set of formulas:

$$
\begin{aligned}
&G(0)/f(0) - F(0)/f(0) = \sum_{n=1}^{\infty} a_n/m_n{}^n = M_0,\\
&G^{(1)}(0)/f^{(1)}(0) - F^{(1)}(0)/f^{(1)}(0) = \sum_{n=1}^{\infty} a_n/m_n{}^{n-1} = M_1,\\
&\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&G^{(j)}(0)/f^{(j)}(0) - F^{(j)}(0)/f^{(j)}(0) = \sum_{n=j}^{\infty} a_n/m_n{}^{n-j} = M_j,\\
&\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
\tag{11}
$$

provided that a_j satisfy the following set of equations:

$$
\begin{aligned}
&m_1 a_1 = F^{(2)}(0)/f^{(2)}(0),\\
&\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&m_1{}^{j-1}a_1 + m_2{}^{j-2}a_2 + \cdots + m_{j-1}\,.\,a_{j-1} = F^{(j)}(0)/f^{(j)}(0),\\
&\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
\tag{12}
$$

and m_j are chosen increasing so rapidly that $|a_j| \leq 1$. We observe then that $M_0, M_1, M_2, \cdots$ form a bounded sequence of numbers. Letting $M_j = N^{(j)}(0)/f^{(j)}(0)$, Theorem I can be applied for the construction of a function $N(x) = \sum_{j=1}^{\infty} \alpha_j f(m_j x)$ having $N^{(j)}(0)$ for initial values at $x=0$. ($N^{(0)}(0) = N(0)$). A function $F(x)$ with any assigned set of initial values $F(0), F^{(j)}(0)$ can be represented by

$$
\begin{aligned}
F(x) = G(x) - N(x) &= F(0) + \sum_{n=1}^{\infty} [(a_n/m_n{}^n) - \alpha_n] f(m^n x)\\
&= F(0) + \sum_{n=1}^{\infty} c_n f(m_n x).
\end{aligned}
\tag{13}
$$

Thus the restriction on the initial values involved in Theorem I has been removed.

We shall consider now the order of approximation furnished by the representation (13); this will include the representation of Theorem I as a particular case. It will be noted that $|a_n| \leq 1$ and that m_j should and can be taken so that $m^{n+1}/m_n > \lambda > 1$. Since $|\alpha_n| < h(m_1 \cdots m_{n-1}/m_n{}^{n-1})$ it will then follow that

$$
|C_n| < h'\left(\frac{m_1 \cdots m_{n-1}}{m_n{}^{n-1}}\right).
\tag{14}
$$

Letting $R_n(x)$ denote the remainder after $(n+1)$ terms in (13) and using (14), we have

$$|R_n(x)| < h'M\left\{\left(\frac{m_1m_2\cdots m_n}{m^n{}_{n+1}}\right)+\left(\frac{m_1m_2\cdots m_{n+1}}{m_{n+2}{}^{n+1}}\right)\right.$$
$$\left.+\cdots\left(\frac{m_1m_2\cdots m_{n+j}}{m_{n+j+1}{}^{n+j}}\right)+\cdots\right\},$$

where $|f(x)| < M$ (for all values of x). Since $m_j/m_k < 1/\lambda < 1$ if $j < k$,

$$|R_n(x)| < h'M\{(1/\lambda^n)+(1/\lambda^{n+1})+\cdots+(1/\lambda^{n+j})+\cdots\} < h/\lambda^n,$$
$$(h = \text{constant}).$$

We have therefore the following theorem:

THEOREM III. *The approximation furnished by the representation of Theorem I and, more generally, by the representation* (13), *is of the order of* $1/\lambda^n$, where $r > 1$.

3. *Representation in a Series of Infinite Integrals.* Let $g(m_jx) = \int_0^\infty \frac{e^{-t}dt}{1+m_jxt}$, $x \geq 0$, $m_j > 0$; differentiation under the sign of integration with respect to x is possible, so that

$$(15) \qquad (d^n/dx^n)\, g(m_jx) = (-1)^n m_j{}^n n! \int_0^\infty \frac{t^n e^{-t}dt}{(1+m_jxt)^{n+1}}.$$

Letting $x = 0$, we derive

$$(d^n/dx^n)\, g(m_jx) = (-1)^n m_j{}^n n! \int_0^\infty t^n e^{-t}dt.$$

Since

$$\int_0^\infty t^n e^{-t}dt = \Gamma(n+1) = n!$$

$$(16) \qquad (d^n/dx^n)\, g(m_jx) = (-1)^n m_j{}^n (n!)^2 \text{ and } g^{(n)}(0) = (-1)^n (n!)^2.$$
$$x=0$$

From (16) it follows that the formal Taylor's expansion of $g(m_jx)$ at $x = 0$ diverges for all x. Also, we have

$$(17) \quad |(d^n/dx^n)\, g(m_jx)| \leq |(d^n/dx^n)\, g(m_jx)| = m_j{}^n (n!)^2 \quad (x \geq 0)$$
$$\text{and } |g^{(n)}(x)| \underset{x=0}{\leq} (n!)^2.$$

Let m_k be positive numbers such that $m_{k+1}/m_k > \lambda > 1$, $(k = 1, 2, \cdots)$ and $F(0)$, $F^{(n)}(0)$ such that $|F(0)|$, $|F^{(n)}(0)/(n!)^2| < G$, $(n = 1, 2, \cdots)$.

Consider

$$(18) \qquad F_n(x) = a_1g(m_1x) + a_2g(m_2x) + \cdots + a_ng(m_nx),$$

where $g(x)=\int_0^\infty \frac{e^{-t}dt}{1+xt}$; determine $a_1, a_2, \cdots, a_n$ by the equations:

$$F(0)=a_1+a_2+\cdots+a_n,$$

$$(-1)F^{(1)}(0)/(1!)^2=m_1a_1+m_2a_2+\cdots+m_na_n,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{19}$$

$$(-1)^{n-1}F^{(n-1)}(0)/[(n-1!)]^2=m_1^{n-1}a_1+\cdots+m_n^{n-1}a_n.$$

These equations differ from the equations (3) in having their left members different; but, as in the other case, the left members are bounded as a set; therefore, coefficients a_k can be obtained, replacing the quantities $F(0)/f(0)$, $F^{(j)}(0)/f^{(j)}(0)$ in the formula (4) by $F(0)$, $(-1)^jF^{(j)}(0)/(j!)^2$; then

$$a_k=(1/2\pi i)\int_C \frac{\phi_n(z^2)\cdot\psi(z)dz}{(z^2-m_k)\phi_n^{(1)}(m_k)}, \tag{20}$$

where $\phi_n(z)=\prod_{j=1}^{n}[1-(z/m_j)]$, $\psi(z)=\sum_{j=0}^{\infty}\frac{(-1)^jF^{(j)}(0)}{(j!)^2}\frac{1}{z^{2j+1}}$ and C is a circle having $z=0$ for the center and a radius $R\geq G$; further,

$$\lim_{n\to\infty}a_k=\alpha_k=(1/2\pi i)\int_C \frac{\phi(z^2)\psi(z)dz}{(z^2-m_k)\phi^{(1)}(m_k)} \tag{21}$$

where $\phi(z^2)=\lim_{n\to\infty}\phi_n(z^2)=\prod_{j=1}^{\infty}[1-(z^2/m_j)]$ and converges, which accounts for the existence of α_k. When $n\to\infty$, $F_n(x)\to F(x)$ and (18) will become

$$F(x)=\alpha_1g(m_1x)+\alpha_2g(m_2x)+\cdots+\alpha_ng(m_nx)+\cdots. \tag{22}$$

This process can be justified as follows:

Since the numbers a_k and α_k satisfy the inequalities (6), it follows that (18) and its successive derivatives converge absolutely and uniformly for all values of n; the same is true of the limiting series (22) (x is restricted to the values $x\geq 0$). Here, however, the function $g(x)$, though possessing the property that $g(x)$, $g^{(n)}(x)$, $(n=1, 2, \cdots)$, are each bounded when $x\geq 0$, does not possess the property of the functions of class (A) (used in the representation of Theorem I) of having all of its successive derivatives bounded as a set. A theorem similar to Theorem I can now be formulated.

THEOREM IV. *Let m_j be a sequence of positive numbers such that for all n*

$$m_{n+1}/m_n>\lambda>1.$$

Let $g(x)=\int_0^\infty \frac{e^{-t}dt}{1+xt}$ and let $F(0)$, $F^{(j)}(0)$, $(j=1, 2, \cdots)$, be num-

bers such that $|F(0)|$, $|F^{(j)}(0)/(j!)^2| < G$, *a positive number. Coefficients* α_j *can then be so determined that the series*

$$F(x)=\alpha_1 g(m_1x)+\alpha_2 g(m_2x)+\cdots+\alpha_n g(m_nx)+\cdots \tag{23}$$

and all its derivatives will converge when $x \geq 0$, *and the sum function will have* $F(0)$, $F^{(1)}(0), \cdots$ *for initial values at* $x=0$. *The coefficients* α_j *can be given by* (21).

DEFINITION. $F(x)=\sum_{j=1}^{\infty}\alpha_j g(m_jx)$ *where* $|\alpha_j| < h\left(\frac{m_1m_2\cdots m_{j-1}}{m_j{}^{j-1}}\right)$ *and* $\lim_{n\to\infty}\left(\frac{m_1m_2\cdots m_n}{m_{n+1}}\right)=0$ *will be said to be of class* (C).

The difference of two functions of class (C) is of class (C); in proving that a function of class (C) with zero initial values vanishes identically, we arrive at the set of equations (9); whence, the proof is exactly the same as that by which Theorem II was derived. Hence, we have a theorem:

THEOREM V. *Functions of class* (C) *are determined uniquely by their initial values at* $x=0$.

In removing the restriction on the initial values $F(0)$, $F^{(j)}(0)$ in Theorem IV, we let $F(0)$, $F^{(j)}(0)$ be any set of numbers and construct a function

$$G(x)=F(0)+\sum_{n=1}^{\infty} a_n\, g(m_nx)/m_n{}^n;$$

differentiating in succession and letting $x=0$ we have two sets of equations differing from (11) and (12) only in having $F(0)$, $(-1)^jF^{(j)}(0)/(j!)^2$ and $G(0)$, $(-1)^j G^{(j)}(0)/(j!)^2$ in place of $F(0)/f(0)$, $F^{(j)}(0)/f^{(j)}(0)$ and $G(0)/f(0)$, $G^{(j)}(0)/f^{(j)}(0)$, respectively. Letting $M_j=(-1)^j N^{(j)}(0)/(j!)^2$ and having M_j bounded, as a set, we construct by the aid of Theorem IV a function $N(x)=\sum_{j=1}^{\infty}\alpha_j g(m_jx)$ having $N(0)$, $N^{(j)}(0)$ for its initial values at $x=0$. Therefore, a function $F(x)$ with any assigned set of initial values $F(0)$, $F^{(j)}(0)$ can be given by

$$F(x)=G(x)-N(x)=F(0)+\sum_{j=1}^{\infty}(a_j/m^j{}_j-\alpha_j)g(m_jx) \tag{24}$$

$$=F(0)+\sum_{n=1}^{\infty} c_n g\ (m_nx).$$

This removes the restriction on initial values in Theorem IV.

We have

$$|c_n| < h'\left(\frac{m_1 \cdots m_{n-1}}{{m_n}^{n-1}}\right). \tag{25}$$

Letting $R_n(x)$ denote the remainder after $(n+1)$ terms in (23) or (24), we have

$$|R_n(x)| < h'\left\{\left(\frac{m_1 m_2 \cdots m_n}{{m^n}_{n+1}}\right)+\cdots\right\}, \text{ since } |g(x)| \leq 1, \ (x \geqq 0).$$

Hence

$$|R_n(x)| < h/\lambda^n, \qquad (\lambda > 1, x \geq 0).$$

Therefore we have

THEOREM VI. *The approximation furnished by the representations in Theorem* IV *and* (24) *is of the order of* $1/\lambda^n$ $(\lambda > 1)$.

Note. As in Theorem III the approximation can be made closer by making the constants m_j increase more rapidly, for then λ is greater.

We have

$$|g^{(n)}(x)| = n! \int_0^\infty \frac{t^n e^{-t} dt}{(1+xt)^{n+1}} < n! \left(\int_0^\infty t^n e^{-t} dt\right).\left(\int_0^\infty dt/(1+xt)^{n+1}\right)$$

$$= (n!)^2 \int_0^\infty dt/(1+xt)^{n-1}.$$

Letting $1+xt = v$ and $x > x_0 > 0$,

$$|g^{(n)}(x)| < (n!)^2 \int_1^\infty \frac{dv}{xv^{n+1}} = \frac{n!(n-1)!}{x} \tag{26}$$

$$\text{and } |g^{(n)}(m_j x)| < \frac{n!(n-1)!}{m_j x}$$

Hence,

$$|F^{(n)}(x)| = \left|\sum_{j=1}^\infty m_j{}^n \alpha_j g^{(n)}(m_j x)\right| < \sum_{j=1}^\infty |m_j \alpha_j g^{(n)}(m_j x)| \tag{27}$$

$$< \frac{n!(n-1)!}{x} \sum_{j=1}^\infty m_j{}^{n-1} |\alpha_j| = P_n/x.$$

The constants P_n are bounded for every finite n on account of the inequalities satisfied by α_j; but as a set they are not necessarily bounded.

From (27) it follows that $\lim_{n\to\infty} F^{(n)}(x) = 0$ (not necessarily uniformly with respect to n).

Consequently, it is shown that *it is possible to construct functions of a real variable with any assigned set of initial values at* $x = 0$, *such that all its*

successive derivatives of finite orders are zero at infinity. All the representations of section (3) of this paper possess this property.

With m_j restricted as before and α_j satisfying inequalities (6), functions of the form

$$F(x)=\sum_{j=1}^{\infty}\alpha_j\int_0^{\infty}\frac{e^{-t}dt}{1+m_jxt} \qquad (x\geq 0) \tag{28}$$

are such that $F(1/x)$ can be represented asymptotically for large values of x; i. e., functions of this section can be so represented.

We observe that

$$\int_0^{\infty}e^{-t}\left(\sum_{j=0}^{\infty}\frac{\alpha_j}{1+m_jxt}\right)dt=\sum_{j=1}^{n}\alpha_j\int_0^{\infty}\frac{e^{-t}dt}{1+m_jxt}+\int_0^{\infty}e^{-t}\left(\sum_{j=n+1}^{\infty}\frac{\alpha_j}{1+m_jxt}\right)dt.$$

Thus

$$\left|\int_0^{\infty}e^{-t}\left(\sum_{j=1}^{\infty}\frac{\alpha_j}{1+m_jxt}\right)dt-\sum_{j=1}^{n}\alpha_j\int_0^{\infty}\frac{e^{-t}dt}{1+m_jxt}\right|=\left|\int_0^{\infty}e^{-t}\left(\sum_{j=n+1}^{\infty}\frac{\alpha_j}{1+m_jxt}\right)dt\right|\leq\{|\alpha_{n+1}|+|\alpha_{n+2}|+\cdots\}=\epsilon_n,$$

where ϵ_n is independent of x and can be made arbitrarily small and $x\geq 0$.

Hence

$$\lim_{n\to\infty}\sum_{j=1}^{n}\alpha_j\int_0^{\infty}\frac{e^{-t}dt}{1+m_jxt}=F(x)=\int_0^{\infty}e^{-t}\left(\sum_{j=1}^{\infty}\frac{\alpha_j}{1+m_jxt}\right)dt. \tag{29}$$

Using this expression for $F(x)$ we integrate by parts n times and obtain

$$F(x)=L(0)+L^{(1)}(0)\cdot x+\cdots+L^{(n)}(0)\cdot x^n+x^{n+1}\int_0^{\infty}e^{-t}L^{(n+1)}(xt)\,dt, \tag{30}$$

where $L(xt)=\sum_{j=1}^{\infty}\frac{\alpha_j}{1+m_jxt}$ and hence $L^{(n)}(0)=(-1)^n n!\sum_{j=1}^{\infty}\alpha_j m_j{}^n$.

Further, it can be verified that

$$|L^{(n+1)}(xt)|\leq(n+1)!\sum_{j=1}^{\infty}|\alpha_j|\,m_j{}^{n+1}=G_{n+1},$$

so that

$$\left|\int_0^{\infty}e^{-t}L^{(n+1)}(xt)\,dt\right|\leq G_{n+1}. \tag{31}$$

On account of the inequalities satisfied by α_j, it follows that the quantities $L(0)$, $L^{(j)}(0)$ are bounded for all finite values of n, but are not necessarily bounded as a set. (If α_j are all positive the constants $L(0)$, $L^{(n)}(0)$ are certainly not bounded as a set). The G_n are unbounded as a set.

Using (30) and (31) we have

$$(32)\quad |F(x)-\{L(0)+L^{(1)}(0)\cdot x+\cdots+L^{(n)}(0)\cdot x^n\}|\leq x^{n+1}\cdot G_{n+1}.$$

Replacing x by $1/x$ we have

$$(33)\quad x^n\,|F(1/x)-\{L(0)+L^{(1)}(0)\cdot 1/x+\cdots+L^{(n)}(0)\cdot 1/x^n\}|\leq G_{n+1}/x.$$

Therefore for large values of x

$$(34)\quad F(1/x)\sim\sum_{n=0}^{\infty}L^{(n)}(0)/x^n=\sum_{n=0}^{\infty}\frac{(-1)^n n!\sum_{j=0}^{\infty}\alpha_j m_j{}^n}{x^n},\quad (L^{(0)}(0)=L(0)),$$

asymptotically; * equation (34) amounts to the statement that for small values of x the function $F(x)$ can be computed from a finite number of terms of the series $\sum_{n=1}^{\infty}L^{(n)}(0)\cdot x^n$.

4. *Representation at an Essentially Singular Point.* We shall have $z=0$ as an essentially singular point; let it be a limiting point of a sequence of poles situated on the positive half of the imaginary axis. The problem will be to construct a function

$$(35)\qquad F(x)=\sum_{j=1}^{\infty}[B_j/(iy_j-x)],$$

where y_j are positive, real and such that $y_n/y_{n+1}>\lambda>1$ for all values of n, with an assigned set of initial values $F(0)$, $F^{(j)}(0)$ such that

$$|F(0)|,\ |F^{(j)}(0)/j!|<G,\qquad (j=1,2,\cdots).$$

Assume $F_n(x)=\sum_{j=1}^{n}[b_j/(iy_j-x)]$ and determine the constants b_j by a set of n equations:

$$\begin{aligned}
&C_0=b_1+b_2+\cdots+b_n,\\
&iF(0)=(b_1/y_1)+(b_2/y_2)+\cdots+(b_n/y_n),\\
&i_2F^{(1)}(0)/1!=(b_1/y_1{}^2)+(b_2/y_2{}^2)+\cdots+(b_n/y_n{}^2),\\
&\qquad\cdots\qquad\cdots\qquad\cdots\qquad\cdots\qquad\cdots\qquad\cdots\qquad\cdots\qquad\cdots\\
&i^{n-1}F^{(n-2)}(0)/(n-2)!=(b_1/y_1{}^{n-1})+(b_2/y_2{}^{n-1})+\cdots+(b_n/y_n{}^{n-1}),
\end{aligned}\qquad (36)$$

* A definition of an asymptotic expansion is given in Whittaker and Watson, Modern Analysis, third edition, Cambridge, 1920, p. 151.

these equations are suggested by those that are derived when in $F(x) = \sum_{j=1}^{\infty} [B_j/(iy_j - x)]$ and its successive derivatives x is put equal to zero; equations (36) reduce to the latter set when $n \to \infty$. The first of the equations (36) is introduced in order to have (36) of the same form as (19) and (3), since then it is known how to solve those sets; C_0 is an arbitrary constant such that $|C_0| < G$. A solution of (36) is given by

$$b_k = (1/2\pi i) \int_C \frac{\phi_n(z^2)\psi(z)\,dz}{[z^2 - (1/y_k)]\phi_n^{(1)}(1/y_k)}, \tag{37}$$

where $\phi_n(z) = \prod_{j=1}^{\infty} (1 - y_j z)$,

$$\psi(z) = C_0(1/z) + iF(0)(1/z^3) + [i^2F^{(1)}(0)/1!]\cdot(1/z^5) + [i^3F^{(2)}(0)/2!]\cdot(1/z^7) + \cdots,$$

and C is a circle having $z = 0$ for the center and a radius $R \geq G$; further,

$$\lim_{n\to\infty} b_k = B_k = (1/2\pi i) \int_C \frac{\phi(z^2)\,\psi(z)\,dz}{[z^2 - (1/y_k)]\cdot\phi^{(1)}(1/y_k)}, \tag{38}$$

where $\phi(z^2) = \lim_{n\to\infty} \phi_n(z^2) = \prod_{j=1}^{\infty} (1 - y_j z^2)$ and converges. When $n \to \infty$, $F_n(x)$ approaches $F(x) = \sum_{j=1}^{\infty} [B_j/(iy_j - x)]$. This process is justified since b_k, B_k satisfy the inequalities (6) with m_j replaced by $(1/y_j)$ and therefore $F_n(x) = \sum_{j=1}^{n} [b_j/(iy_j - x)]$ and its successive derivatives converge uniformly for all values of n, the same holding true for $F(x) = \sum_{j=1}^{\infty} [B_j/(iy_j - x)]$. We formulate then the following theorem:

THEOREM VII. *Let y_j be a sequence of positive numbers such that for all values of n*

$$y_n/y_{n+1} > \lambda > 1.$$

Let $F(0)$, $F^{(j)}(0)$ $(j = 1, 2, \cdots)$ be numbers such that $|F(0)|$, $|F^{(j)}(0)/j!| < G$, a positive number. Coefficients B_j can then be so determined that the series

$$F(x) = [B_1/(iy_1 - x)] + [B_2/(iy_2 - x)] + [(B_3/(iy_3 - x)] + \cdots + [B_j/(iy_j - x)] + \cdots \tag{39}$$

(for which $x = 0$ is an essentially singular point, being a limiting point of a sequence of poles) and all its derivatives will converge for all real values of x and the sum function will have $F(0)$, $F^{(1)}(0), \cdots$ for its initial values at $x = 0$. The coefficients B_j are given by (38).

Take two functions with the same initial values, but in their representation by means of Theorem VII let C_0 have different values in each of them; let them have the same poles. The difference of such two functions

$$A(x) = [\gamma_1/(iy_1 - x)] + [\gamma_2/(iy_2 - x)] + \cdots \tag{40}$$

has zero initial values; making use of (38) we derive

$$\gamma_k = (s/2\pi i) \int_\sigma \frac{\phi(z^2)\,dz}{z[z^2 - (1/y_k)]\,\phi^{(1)}(1/y_k)} \tag{41}$$

where s is a constant and $\phi(z) = \prod_{j=1}^{\infty} (1 - y_j z)$.

Thus, it has been shown, that *since y_j can be chosen in an indefinitely great number of ways, there are infinitely many functions of this class with zero initial values at an essentially singular point.*

Let now $F(0)$, $F^{(j)}(0)$ $(j = 1, 2, \cdots)$ be any set of initial values; we construct a function

$$G(x) = F(0) + \sum_{n=1}^{\infty} [a_n y_n{}^n/(iy_n - x)]; \tag{42}$$

differentiating successively, and letting $x = 0$, we have two sets of equations differing from (11) and (12) by having $iG(0), iF(0)$ and $i^j G^{(j-1)}(0)/(j-1)!$ $i^j F^{(j-1)}(0)/(j-1)!$ and $1/y_j$ in place of $G^{(1)}(0)/f^{(1)}(0)$, $F^{(1)}(0)/f^{(1)}(0)$, $G^{(j)}(0)/f^{(j)}(0)$, $F^{(j)}(0)/f^{(j)}(0)$ and m_j, respectively; letting

$$M_j = i^j N^{(j-1)}(0)/(j-1)!, \qquad (j = 1, 2, \cdots)$$

by Theorem VII we can construct a function $N(x) = \sum_{j=1}^{\infty} [B_j/iy_j - x)]$ with $N^{(0)}(0) = N(0)$, $N^{(j)}(0)$ for initial values. The function

$$F(x) = G(x) - N(x) \tag{43}$$

will have $F(0)$, $F^{(1)}(0)$, $F^{(2)}(0), \cdots$ for initial values at $x = 0$. In the preceding, the first of the formulas (11) is omitted. We have shown therefore that *functions can be constructed having any assigned set of constants for initial values at an essentially singular point.* In the general case y_j must be diminishing rapidly enough in order that the modified set of equations (12) should define the constants a_n of (42) so that, for all values of n, $|a_n| \leq 1$.

The method of this section can be readily generalized. Let $z = 0$ be the limiting point of a sequence of poles γ_j, all situated on a straight line through the origin making an angle $\theta (\neq 0, \neq \pi)$ with the positive half of the real axis; such poles can be given by $\gamma_j = (ctg\theta + i)y_j$; we shall suppose that y_j satisfy the same conditions as before. We consider functions of the form

$$(44) \qquad F(x) = \sum_{j=1}^{\infty} \alpha_j / [(ctg\theta + i) y_j - x];$$

all of the preceding part of this section will hold with respect to functions of the form (44), provided $i^{n-1}F^{n-2}(0)/(n-2)!$ is everywhere replaced by $(ctg\theta + i)^{n-1}F^{n-2}(0)/(n-2)!$ Then in Theorem VII the condition will be introduced that

$$(45) \qquad |(ctg\theta + i)^{n-1} \cdot F^{n-2}(0)/(n-2)!| < G, \qquad (n = 1, 2, \cdots).$$

From (45) it follows that $F(0)$, $F^{(j)}(0)$ are more restricted for values of θ nearer to 0 or π. The restriction on the $F(0)$, $F^{(j)}(0)$ is least in the particular cases when $\theta = \pm (\pi/2)$.

By initial values in the above representations are meant the limits attained by the function and its successive derivatives when the variable approaches the essentially singular point (of the type considered) along a fixed direction. Cauchy's example

$$e^{-1/z^2}$$

has a different type of an essentially singular point at $z = 0$, but the function and its successive derivatives possess definite limits when the variable approaches $z = 0$ along the real axis.

5. *An Example.* The coefficients α_n of the representations considered in Theorems I, IV and VII are of the same general form. Let us consider the case, which could be used in connection with any of those theorems, when m_r or $1/y_r = \lambda^r$ $(\lambda > 1)$; denote the coefficients of $\psi(z)$ by G_j; they will be assumed such that $\psi(z) = \sum_{j=0}^{\infty} (G_j/z^{2j+1})$ is an entire function. It is known how to expand a function $\phi(z) = \prod_{j=1}^{\infty} [1 - (z/\lambda^j)]$ in a power series in z; taking successive derivatives of $\phi(z/\lambda) \cdot [1 - (z/\lambda)] = \phi(z)$ and letting $z = 0$ we derive

$$\phi^{(r)}(0) = (-1)^r r!/(\lambda - 1)(\lambda^2 - 1) \cdots (\lambda^r - 1); \qquad \text{hence}$$

$$\phi(z) = \sum_{r=0}^{\infty} c_r z^r, \qquad c_0 = 1,$$

$$c_r = (-1)^r/(\lambda - 1)(\lambda^2 - 1) \cdots (\lambda^r - 1), \qquad (r = 1, 2, \cdots).$$

Further

$$\phi(z^2)/(z^2 - \lambda^n) = \sum_{r=0}^{\infty} [c_r z^{2r}/(z^2 - \lambda^n)] = \sum_{r=0}^{\infty} {}_nd_r \cdot z^{2r};$$

$$\sum_{r=0}^{\infty} c_r z^{2r} = (z^2 - \lambda^n) \sum_{r=0}^{\infty} {}_nd_r \cdot z^{2r}$$

$$= -{}_nd_0 \lambda^n + \sum_{r=1}^{\infty} ({}_nd_{r-1} - {}_nd_r \lambda^n) z^{2r}.$$

Comparing coefficients, have

$$ {}_nd_{r-1} - {}_nd_r\,\lambda^n = c_r, \quad -{}_nd_0\,\lambda^n = 1. \qquad (r = 1, 2, \cdots), $$

$$ {}_nd_r = -(c_r/\lambda^n) - (c_{r-1}/\lambda^{2n}) - \cdots - (1/\lambda^{(r+1)n}) \qquad (r = 0, 1, 2, \cdots). $$

Consider

$$ \phi^{(1)}(\lambda^n)\cdot\alpha_n = (1/2\pi i)\int_C \frac{\phi(z^2)}{(z^2-\lambda^n)}\cdot\psi(z)\,dz = $$

$$ = (1/2\pi i)\int_0 \Big(\sum_{r=0}^{\infty} {}_nd_r\, z^{2r}\Big)\Big[\sum_{j=0}^{\infty}(G_j/^{2j+1})\Big]dz; $$

both of the power series of the integrand are entire functions and multiplication termwise is justifiable. Since the coefficient of $1/z$ in the integrand is $\sum_{r=0}^{\infty} G_r\,{}_nd_r$, and $\phi^{(1)}(\lambda^n) = \sum_{r=1}^{\infty} r\,c_r\,\lambda^{n(r-1)}$, we obtain finally,

$$ \alpha_n = \Big(\sum_{r=0}^{\infty} G_r\cdot{}_nd_r \Big/ \sum_{r=1}^{\infty} r\,c_r\lambda^{n(r-1)}\Big), \qquad (n = 1, 2, 3, \cdots), \tag{46} $$

where

$$ c_r = (-1)^r/(\lambda-1)(\lambda^2-1)\cdots(\lambda^r-1), \qquad c_0 = 1, $$

$$ {}_nd_r = -(c_r/\lambda^n) - (c_{r-1}/\lambda^{2n}) - \cdots - (1/\lambda^{(r+1)n}). $$

In the numerator of (46) terms can be so rearranged as to have a power series in $1/\lambda^n$ with coefficients independent of n; this would facilitate actual computation.

6. *Application of Darboux's Formula.** Suppose that $F(x)$ is a function of a real variable, indefinitely differentiable, and such that the function and its successive derivatives of all finite orders are finite and continuous on the interval $(0, x)$, then, even though $F(x)$ is not necessarily analytic, Darboux's formula holds; this becomes evident when we follow the steps by means of which that formula is proved.

Let $\phi(t)$ be a polynomial of degree n, $(0 \le t \le 1)$. Consider the identity

$$ (d/dt)\sum_{m=1}^{\infty}(-1)^m\cdot x^m\phi^{(n-m)}(t)\cdot F^{(m)}(xt) $$

$$ = -x\phi^{(n)}(t)F^{(1)}(xt) + (-1)^n x^{n+1}\phi(t)F^{(n+1)}(x\cdot t); $$

integration with respect to t between the limits 0 and 1 gives the formula

$$ \phi^{(n)}(0)[F(x) - F(0)] $$

$$ = \sum_{m=1}^{n}(-1)^{m-1}x^m[\phi^{(n-m)}(1)F^{(m)}(x) - \phi^{(n-m)}(0)F^{(m)}(0)] $$

$$ + (-1)^n x^{n+1}\int_0^1 \phi(t)F^{(n+1)}(xt)\,dt. $$

* *Journal de Mathématiques*, Ser. 3, Vol. 2 (1876), p. 271; also see Whittaker and Watson, *Modern Analysis*, third edition, Cambridge, 1920, p. 125.

In particular, if $\phi(t)=(t-1)^n$ *we have*

$$(47)\qquad F(x)=[\sum_{m=0}^{n}(x^m F^{(m)}(0)/m!)]+[(-1)^n x^{n+1}/n!]\int_0^1 (t-1)^n F^{(n+1)}(xt)\,dt.$$

When $\lim_{n\to\infty}(1/n!)\int_0^1 (t-1)^n F^{(n+1)}(xt)\,dt=0$, $F(x)$ is analytic. With the hypothesis made concerning $F(x)$, it cannot be asserted that $\lim_{n\to\infty}(1/n!)\int_0^1 (t-1)^n F^{(n+1)}(xt)\,dt=0$, even if $\sum_{m=0}^{\infty} x^m F^{(m)}(0)/m!$ converges.

Replacing in (47) x by $1/x$ we have:

$$(48)\qquad |x^m\{F(1/x)-\sum_{m=0}^{n}[(F^{(m)}(0)/m!)\cdot(1/x^m)]\}| = |(1/xn!)\int_0^1 (t-1)^n\cdot F^{(n+1)}(t/x)\,dt|$$

$\leqq [P_{n+1}/x(n+1)!]$ $(P_n \geq |F^{(n)}(x)|$ for all real x and $n=1, 2, \cdots)$.

Hence the following theorem follows:

THEOREM VIII. *Let $F(x)$ be an indefinitely differentiable, non- analytic function such that $F(x)$, $F^{(1)}(x)$, $F^{(2)}(x), \cdots$ are continuous and finite in the interval $(0, x)$; let $F(0)$, $F^{(1)}(0)$, $F^{(2)}(0), \cdots$ be the initial values at $x=0$; then for large values of x*

$$F(1/x)\sim\sum_{m=0}^{\infty}(F^{(m)}(0)/m!)\cdot 1/x^m \text{ asymptotically.}$$

COROLLARY. *If a function $F(x)$ with zero initial values at $x=0$ satisfies the conditions of Theorem VIII then for large values of x*

$$F(1/x)\sim 0 \text{ asymptotically.}$$

These results can be applied to the representations of section (2).

7. *General Observations.* The methods of sections 3 and 4 have an advantage over that of section 2 in having the initial values less restricted (Theorems IV and VII), which advantage would be of importance when m_j (such that $m_{j+1}/m_j>\lambda>1$) or y_j (such that $y_j/y_{j+1}>\lambda>1$) are fixed, since when the restrictions on the initial values are removed it may be necessary to have m_j increasing more rapidly, or y_j decreasing more slowly (to satisfy equations (12) or the modified sets of equations (12)). The method of section 2 offers a greater multiplicity of representations, since there are infinitely many functions of class (A).

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The Abstract Identity of Modular Systems and Ideals.

By Duncan Harkin.

Introduction. The object of this paper is to show the abstract identity of two theories: one, the Theory of Ideals, first developed by Richard Dedekind in 1871; the other, Leopold Kronecker's Modular Systems, made known in the famous "Festschrift" of 1882. The method followed is one of abstraction and transformation by formal equivalence,* in the spirit of E. H. Moore's "General Analysis".

The terminology of the paper agrees closely with that of Kronecker. The theorems, however, follow Dedekind for the most part. By this dual scheme, where the notation and external form is apparently that of Kronecker but where the root idea and fundamental meaning is taken almost bodily from Dedekind, the two theories are seen to be two instances of one and the same general theory.

1. A field $\mathfrak{F}$ is defined as usual.† Given a set of numbers $[a]$ and two fundamental operations S and P (called addition and multiplication), such that either operating on a pair of numbers produces an unique number of the set, and if O stands for either S or P (disjunctively), the field properties may be briefly characterized by the four postulates 1.01-1.04.

1.01 $$O(a_1, a_2) = a = O(a_2, a_1)$$

The operation is commutative.

1.02 $$O(O(a_1, a_2), a_3) = O(a_1, O(a_2, a_3))$$

The operation is associative.

$$P(S(a_1, a_2), a_3) = S(P(a_1, a_3), P(a_2, a_3))$$

Multiplication is distributive with respect to addition.

* E. T. Bell, *Algebraic Arithmetic*, American Mathematical Society Colloquium Publications, Vol. 7 (1927), Chap. V, §§ 1-2.

† E. T. Bell, *Algebraic Arithmetic*, pp. 6-7; L. E. Dickson, *Algebras and their Arithmetics*, Chicago, 1923, p. 201; G. Scorza, *Corpi Numerici e Algebre*, Palermo, 1921, pp. 3-13.

1.03. There exists in the set an identity element i such that, for any number a of the set,

$$O(a, i) = a.$$

1.04. For every number a of the set there exists in the set an inverse r, such that

$$O(a, r) = i.$$

1.1. From these postulates the uniqueness of the identity and inverse elements may be proved. The unique identity element under S is written 0, and, under P, 1. It is also easily shown that cancellation of like components from the two sides of an equality is permissible, i. e. if $O(a_1, a_2) = O(a_1, a_3)$, then $a_2 = a_3$.

1.2. A special variety * is produced by modification of the postulates. Whole numbers are defined by so restricting postulate 1.04 that under multiplication the only number having an inverse is the identity 1 itself. A further restriction of 1.04, so that there is no inverse under either addition or multiplication, except the identity, defines the set of positive whole numbers.

1.3. The indeterminate or arbitrary number u, due to Gauss but first taking important place in the work of Kronecker, is defined as *any* number, in the sense that its having once stood for a particular number a_1 does not prevent it from standing for another number, distinct or the same. This definition of indeterminates leads immediately to the theorem:

1.31. *In an equation in n indeterminataes* (*or powers of the same indeterminate*), *coefficients of like indeterminates* (*or like powers of the same indeterminate*) *are equal.*

By hypothesis, we have

$$\sum_{i}^{1 \cdots n} a_{1i}u_i = \sum_{i}^{1 \cdots n} a_{2i}u_i.$$

Since the u_i are arbitrary, this equation must hold when $u_1 \neq 0$ and $u_i = 0$ $(i = 2, 3, \cdots, n)$; therefore

$$a_{11} = a_{21}.$$

Similarly, for $u_2 \neq 0$, $u_i = 0$ $(i = 1, 3, \cdots, n)$,

$$a_{12} = a_{22};$$

* E. T. Bell, *loc. cit.*, p. 5.

and so on. In general,

$$a_{1i} = a_{2i} \qquad (i = 1, 2, \cdots, n).$$

The alternate form of the theorem is proved in like manner.

1.4. Further elements l_i are adjoined, subject to no other restriction than that of combining in the usual way under S and P. An integrity-domain $\mathfrak{J}$ is then defined as the set $[\psi(l_i; a_j)]$ of all rational integral functions of these elements with whole-number coefficients. That $\mathfrak{J}$ is properly called an "integrity-domain" is seen by observing that the ψ_i satisfy all the postulates for whole numbers. A rationality-domain $\mathfrak{R}$ consists of the set $[\phi]$ of all rational functions of the elements l_i with whole-number coefficients. Consequently, every ϕ can be written as a quotient ψ_1/ψ_2, and the set satisfies the postulates 1.01-1.04.

1.5. Following the nomenclature of Kronecker * and, before him, Gauss, who first introduced into mathematics the terminology of the natural sciences, we define class, genus, and species. A set of n algebraic quantities (definition) λ_i $(i = 1, 2, \cdots, n)$ are said to constitute a class $\mathfrak{C}$ of algebraic quantities if their elementary symmetric functions are equal to quantities of a given rationality-domain $\mathfrak{R}$. If the domain is restricted to an integrity-domain $\mathfrak{J}$, then the class $\mathfrak{C}$ consists of algebraic integers. The n algebraic quantities λ (or integers α) are called conjugate algebraic quantities (or integers).

1.51. *If a class $\mathfrak{C}$ of n conjugate algebraic integers α_i $(i = 1, 2, \cdots, n)$ satisfy the system of equations*

$$(1) \qquad \sum \alpha_1 = \mathfrak{f}_1, \quad \sum \alpha_1\alpha_2 = \mathfrak{f}_2, \cdots, \quad \alpha_1\alpha_2 \cdots \alpha_n = \mathfrak{f}_n,$$

where the $\mathfrak{f}_i$ $(i = 1, 2, \cdots, n)$ belong to a given integrity-domain $\mathfrak{J}$, then any algebraic integer $\alpha = \alpha_i$ $(i = 1, 2, \cdots, n)$ of the class satisfies the equation

$$(2) \qquad \alpha^n - \mathfrak{f}_1\alpha^{n-1} + \mathfrak{f}_2\alpha^{n-2} - \cdots \pm \mathfrak{f}_n = 0$$

or

$$(2') \qquad \alpha^n - \alpha^{n-1}\sum \alpha_1 + \alpha^{n-2}\sum \alpha_1\alpha_2 - \cdots \pm \alpha_1\alpha_2 \cdots \alpha_n = 0;$$

and this last is equivalent to the equation

$$(3) \qquad \prod_4^{1 \cdots n} (\alpha - \alpha_i) = 0.$$

* L. Kronecker, "Grundzüge einer arithmetischen Theorie der algebraischen Grössen", *Crelle's Journal für Mathematik*, Vol. 92 (1882), pp. 1-122; *Werke*, II, 237-387.

We proceed first with the demonstration of the first part of the theorem. By hypothesis,

$$\sum_{i}^{1 \ldots n} \alpha_i - \mathfrak{f}_1 = 0.$$

Multiplying this by α_1, we get

$$\alpha_1 \sum_{i}^{1 \ldots n} \alpha_i - \mathfrak{f}_1 \alpha_1 = 0,$$

which can also be written

$$\alpha_1^2 - \mathfrak{f}_1 \alpha_1 + \sum_{i}^{2 \ldots n} \alpha_1 \alpha_i = 0.$$

Since

$$\mathfrak{f}_2 = \alpha_1 \alpha_2 + \sum_{i,j}^{2 \ldots n} \alpha_i \alpha_j$$

this last equation can finally be put in the form

$$\alpha_1^2 - \mathfrak{f}_1 \alpha_1 + \mathfrak{f}_2 - \sum_{i,j}^{2 \ldots n} \alpha_i \alpha_j = 0.$$

Again, multiplication by α_1 gives

$$\alpha_1^3 - \mathfrak{f}_1 \alpha_1^2 + \mathfrak{f}_2 \alpha_1 - \sum_{i,j}^{2 \ldots n} \alpha_1 \alpha_i \alpha_j = 0,$$

which, as before, can be put

$$\alpha_1^3 - \mathfrak{f}_1 \alpha_1^2 + \mathfrak{f}_2 \alpha_1 - \mathfrak{f}_3 + \sum_{i,j,k}^{2 \ldots n} \alpha_i \alpha_j \alpha_k = 0.$$

. . . . . .

This easily leads to the empirical formula

$$\sum_{i}^{1 \ldots r} (-1)^i \mathfrak{f}_{r-1} \alpha_1^i + (-1)^r \sum_{i, \ldots, k}^{2 \ldots n} \alpha_1 \alpha_i \cdots \alpha_k = 0,$$

which is proved by mathematical induction up to, as the notation indicates, $r = n$. For $r = n$,

$$\sum_{i}^{1 \ldots n} (-1)^i \mathfrak{f}_{n-i} \alpha_1^i + (-1)^n \sum_{i, \ldots, k}^{2 \ldots n} \alpha_1 \alpha_i \cdots \alpha_k = 0.$$

This is obviously the same as

$$\sum_{i}^{1 \ldots n} (-1)^i \mathfrak{f}_{n-i} \alpha_1^i + (-1)^n \alpha_1 \alpha_2 \cdots \alpha_n = 0$$

or

$$\sum_{i}^{1 \ldots n} (-1)^i \mathfrak{f}_{n-i} \alpha_1^i + (-1)^n \mathfrak{f}_n = 0,$$

which can be written

$$\sum_{i}^{0 \ldots n} (-1)^i \mathfrak{f}_{n-i} \alpha_1^i = 0.$$

By parity,

$$\sum_{i}^{0 \cdots n} (-1)^i \mathfrak{f}_{n-i} \alpha_j{}^i = 0 \qquad (j = 1, 2, \cdots, n)$$

or, if $\alpha = \alpha_j$ $(j = 1, 2, \cdots, n)$,

$$\sum_{i}^{0 \cdots n} (-1)^i \mathfrak{f}_{n-i} \alpha^i = 0,$$

as was to be proved.

The last part of the theorem is proved by expanding the product

$$\prod_{i}^{1 \cdots n} (\alpha - \alpha_i) = 0.$$

This gives the equation

$$\alpha^n - \alpha^{n-1} \sum \alpha_i + \alpha^{n-2} \sum \alpha_i \alpha_j - \cdots \pm \alpha_1 \alpha_2 \cdots \alpha_n = 0.$$

By means of the Galois theory of equations, equations (1) and (2) may be shown equivalent.

1.52. *The sum, difference, or product of any two algebraic integers α_1 and α_2 of a class $\mathfrak{C}$ is an algebraic integer of the same class.*

1.6. A genus $\mathfrak{G}$ is composed of all rational functions of the algebraic quantities of a given class $\mathfrak{C}$ under a rationality-domain $\mathfrak{R}$. Under a given integrity-domain, $\mathfrak{J}$, all the rational integral functions of the algebraic integers belonging to a class $\mathfrak{C}$ constitute a species $\mathfrak{S}$. Given an integrity domain and, under it, a class $\mathfrak{C}$ of algebraic integers and a genus $\mathfrak{G}$, then a principal species $\mathfrak{S}^*$ is defined as the totality of all the rational integral functions of the integral algebraic numbers which (functions) also belong to the genus $\mathfrak{G}$.

1.61. *For every principal species $\mathfrak{S}^*$ there exists a finite basis $\mathfrak{B}(\omega_1, \omega_2, \cdots, \omega_n)$ of $\mathfrak{S}^*$ such that any quantity of the principal species $\mathfrak{S}^*$ may be expressed linearly in terms of the elements ω_i of the basis with integral quantities ψ_i of the integrity domain $\mathfrak{J}$ for coefficients.*

1.7. Conjugate algebraic units are algebraic integers whose product is unity. This is equivalent to saying that the norm of an algebraic unit ϵ is 1, or that an algebraic unit is an algebraic integer satisfying an equation whose first and last coefficients are both 1.

2. *Forms and their Properties.*

2.1. A form F is a linear combination of algebraic integers, belonging

to a given species and genus, with indeterminate coefficients. More generally, F may be regarded as a form in the indeterminates with integral algebraic coefficients. Thus

$$F = \sum u_i\alpha_i = \sum \alpha_i u_i \qquad (i = 1, 2, \cdots, n).$$

A form F is said to contain an algebraic integer α if α can be expressed linearly in terms of the elements of the form. More generally, a form F_1 is said to contain a form F_2 if each element of the second form can be expressed linearly in terms of the elements of the first form.

2.11. *A form* $F = \sum u_i\alpha_i$ $(i = 1, 2, \cdots, n)$ *contains each of its elements* α_i $(i = 1, 2, \cdots, n)$.

2.12. *Every form is equivalent to itself.*

2.121. *A form remains unaltered by the addition to its elements of any quantity contained in the form.*

By definition,

$$F = u_1\alpha_1 + u_2\alpha_2 + \cdots + u_n\alpha_n.$$

Consider

$$F' = u_1'\alpha_1 + u_1'\psi\alpha_2 + u_2'\alpha_2 + \cdots + u_n'\alpha_n.$$

This may be written

$$F' = u_1'\alpha_1 + (u_1'\psi + u_2')\alpha_2 + \cdots + u_n'\alpha_n$$

or

$$F' = u_1\alpha_1 + u_2\alpha_2 + \cdots + u_n\alpha_n,$$

where

$$u_2 = u_1'\psi + u_2' \quad \text{and} \quad u_i = u_i' \qquad (i = 1, 3, 4, \cdots, n).$$

2.13. *The sum of two forms is a form.*

By definition,

$$F_1 = \sum_i^{1 \cdots r} u_i\alpha_i' \quad \text{and} \quad F_2 = \sum_i^{1 \cdots s} v_i\alpha_i''$$

$$\therefore \quad F_1 + F_2 = \sum_i^{1 \cdots n} w_i\alpha_i,$$

where $n = r + s$, and $w_i\alpha_i = u_i\alpha_i'$ $(i = 1, \cdots, r)$ or $w_i\alpha_i = v_i\alpha_i''$ $(i = r + 1, \cdots, s)$.

2.14. *The product of two forms is a form.*

Defining F_1 and F_2 as in (2.13), we have

$$F_1F_2 = \sum w_i\alpha_i, \qquad (i = 1, \cdots, n)$$

where $n = rs$, and $w_i\alpha_i = u_hv_k\alpha_h'\alpha_k''$ $(h = 1, \cdots, r;\ k = 1, \cdots, s)$. This substitution is permissible by 1.52 and the fact that the product of two indeterminates is again indeterminate.

2.15. *The sum of two forms contains either form.*

2.16. *The product of two forms is contained in either of the components.*

F_1 and F_2 being defined as in 2.13 and their product F_1F_2 as in 2.14, it is sufficient to show that F_1 contains F_1F_2. To show this, every element $\alpha_i'\alpha_j''$ of F_1F_2 must be linearly expressible in terms of the elements α_i' of F_1, i.e. $\alpha_i'\alpha_j'' = \sum u_i'\alpha_i'$ $(i = 1, \cdots, r)$. This is possible by putting $u_i' = w_{ij}\alpha_j''$.

2.17. *Given two forms F_1 and F_2, then, if there exists a form F such that the product F_2F is equivalent to F_1, it follows that F_2 contains F_1.*

By 2.16, F_2 contains F_2F which, by hypothesis, is equivalent to F_1.

The converse of this theorem, namely: *If F_1 is contained in F_2, there exists an unique form F such that F_1 is equivalent to F_2F,* is not evident. Its validity will be the aim and the conclusion of the present section.

2.18. *If a form F is contained in a quantity ψ of the original domain, then there exists a form F' such that $F = \psi F'$.*

By hypothesis, ψ contains $F = \sum u_i\alpha_i$ and therefore contains each α_i, i.e. $\alpha_i = \psi\alpha_i'$. Since $F = \sum u_i\alpha_i = \psi \sum u_i\alpha_i'$, it is evident that $F' = \sum u_i\alpha_i'$.

2.19. *If $\psi F_1 = \psi F_2$, then $F_1 = F_2$.*

2.2. The norm Γ of a form F is the product of F and all forms conjugate to F.

2.21. *The elements of the norm Γ of a form F are rational integral quantities.*

By definition,

$$NF = \Gamma = \prod_{j}^{1 \cdots n} (u_1\alpha_{1j} + u_2\alpha_{2j} + \cdots + u_n\alpha_{nj}).$$

Since this product expanded involves only symmetric functions of the α_{ij},

it follows by 1.51 that the resulting coefficients will be quantities of the original field.

The reduced norm T of a form F is the quotient of the norm Γ of the form divided by the G. C. D. d of its rational integral coefficients, i. e. $T = \Gamma/d$.

2.22. *The norm of the product of two forms is equivalent to the product of the norms of the forms.*

Write $NF_1 = F_1\bar{F}_1$, where $\bar{F}_1 = F_{12}F_{13}\cdots F_{1n}$, the F_{1j} $(j = 1, 2, \cdots, n)$ being conjugate forms. Then

$$NF_1NF_2 = F_1\bar{F}_1F_2\bar{F}_2 = F_1F_2\bar{F}_1\bar{F}_2 = F\bar{F},$$

where $F = F_{11}F_{21}$ and $\bar{F} = F_{12}F_{21}F_{13}F_{23}\cdots F_{1n}F_{2n}$.

2.3. A primitive form E (Einheitsform) is a form whose norm is equivalent to 1.

2.31. *Unity* 1 *contains every form.*

2.32. *A primitive form E contains every form F.*

By 2.16, E contains its norm NE which, by definition, is equivalent to 1. But 1, by 2.31, contains any form F. $\therefore$ E contains F.

2.33. *Every primitive form E is equivalent to* 1.

By the proof of 2.32, E contains 1 and, by 2.31, 1 contains E.

2.34. *The sum of two primitive forms E_1 and E_2 is a primitive form.*

By 2.13, $E_1 + E_2$ is equivalent to some form F and is therefore contained in F. But $E_1 + E_2$ contains E_1, by 2.15. Hence F contains E_1 and, by 2.22, NF contains $NE_1 = 1$. Since NF is also contained in 1, by 2.31, $NF = 1$ and F is primitive.

2.35. *The product of two primitive forms is equivalent to a primitive form.*

By 2.14, E_1E_2 is equivalent to some form, say F; and, by 2.22, $NE_1NE_2 = NF$. By definition, $NE_1 = 1$ and $NE_2 = 1$, therefore $NF = 1$ and so F is primitive.

2.36. *The sum of a primitive form E and of any form F is equivalent to the primitive form E.*

The sum $E + F$ contains E, by 2.15. It remains now to show that E contains $E + F$. But E contains itself and, by 2.32, E contains F; therefore E contains $E + F$.

2.37. *The product of a primitive form E and any form F is equivalent to F.*

F contains EF, by 2.16. It now remains to show that EF contains F. This follows from 2.12 and 2.33.

These theorems justify Dedekind's definition of an ideal and bear witness to his "remarkably sharp-witted method of deduction",* for in the two properties which he makes definitive lies the germ of the whole development. Dedekind's definition † can be summarily put

$$\text{I} \quad S(\mathfrak{a}, \mathfrak{a}) = \mathfrak{a}$$
$$\text{II} \quad P(\mathfrak{o}, \mathfrak{a}) = \mathfrak{a}$$

or, in the notation here used,

$$S(F, F) = F$$
$$P(E, F) = F.$$

2.38. *If $F_1 + F_2$ is equivalent to a primitive form E and if F_1 contains the product F_2F_3, then F_1 contains F_3.*

By hypothesis, F_1 contains F_2F_3, and, by 2.16, F_1 contains F_1F_3; therefore F_1 contains $F_1F_3 + F_2F_3 = F_3(F_1 + F_2)$ which, by hypothesis, equals F_3E. But, by 2.37, $F_3E = F_3$, whence the theorem.

2.39. *The set of all the forms which can contain a given form is finite.*

By 2.21, the norm of the given form F is a rational quantity ψ, which, by 2.16, is contained in F. Let $F' = \sum u_i\alpha_i$ $(i = 1, 2, \cdots, n)$ be any form containing F. By 1.81,

$$F' = u_1(\psi_{11}\omega_1 + \psi_{12}\omega_2 + \cdots + \psi_{1n}) + \cdots + u_n(\psi_{n1}\omega_1 + \cdots + \psi_{nn}\omega_n),$$

where $(\omega_1, \omega_2, \cdots, \omega_n)$ is a basis. By 2.121, since F' contains ψ, we may put

$$F' = u_1'(\psi_{11}\omega_1 + \cdots + \psi_{1n}\omega_n) + \cdots + u_n' \ (\psi_{n1}\omega_1 + \cdots + \psi_{nn}\omega_n) + \psi$$
$$= u_1'(\psi_{11}'\omega_1 + \cdots + \psi_{1n}'\omega_n) + \cdots + u_n' \ (\psi_{n1}'\omega_1 + \cdots + \psi_{nn}'\omega_n) + \psi,$$

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* Kronecker, *Grundzüge*, § 21.

† R. Dedekind, "Sur la théorie des nombres entiers algébriques", *Bulletin des Sciences Mathématiques et Astronomiques*, t. 9 (1876), pp. 278-288.

where ψ_{ij}' are the residues (mod. ψ) of the ψ_{ij}. But, since the ψ_{ij} are rational quantities, the number of residues (mod. ψ) is finite. Hence the theorem.

2.4. A prime form P is a non-primitive form such that, if any form F' contains P, then F' is equivalent (disjunctively) either to P or to a primitive form E.

2.41. *The sum of two prime forms P_1 and P_2 is equivalent to a primitive form E or else (disjunctive) the two prime forms are equivalent to each other.*

The sum $P_1 + P_2$ is equivalent, by 2.13, to some form F which therefore contains $P_1 + P_2$ and, consequently, by 2.15, each of P_1 and P_2 separately. By the definition of a prime form, this leaves two alternatives: either F is equivalent to both P_1 and P_2 (hence P_1 equivalent to P_2) or F is a primitive form E.

2.42. *Every non-primitive form F is contained in some prime form P.*

Either F is already a prime form (and the condition immediately satisfied) or, by 2.39, F is contained in a finite set of forms. But among this finite set there is some form, say F_h, of smallest norm. Hence the prime form P sought is F_h.

2.43. *For every form F there exists a form F' such that their product FF' is a quantity of the original field.*

This is satisfied by choosing $F' = \bar{F}$, where $F\bar{F} = NF$.

This theorem is the crux of the theory. Its importance is pointed out by both Dedekind and Kronecker, as well as by subsequent writers.*

2.44. *If $F_1 = F_1F_2$, then F_2 is primitive.*

By hypothesis, $F_1 = F_1F_2$; but, by 2.43, there exists an F' such that $F_1F' = \psi$. Therefore $\psi = \psi F_2$, and so $F_2 = E$, by 2.19 and 2.33.

2.45. *If $F_1F_2 = F_1F_3$, then $F_2 = F_3$.*

By hypothesis, $F_1F_2 = F_1F_3$ and, by 2.43, there is a form F'_1 so that $F'_1F_1F_2 = F'_1F_1F_3$ gives $\psi F_2 = \psi F_3$ which, by 2.19, implies $F_2 = F_3$.

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* See, in particular, A. Hurwitz, "Ueber die Theorie der Ideale," *Göttinger Nachrichten* (1894), pp. 291-298.

2.46. *If a form F_1 is contained in a prime form P, then there exists an unique form F_2 such that $F_1 = F_2P$.*

By hypothesis, F_1 is contained in P, and so F_1F' in PF'. But, by 2.43, F' can be chosen so that $PF' = \psi$, and 2.18 implies an F_2 so that $F_1F' = \psi F_2$. From this, $F_1F'P = \psi F_2P$ and therefore $\psi F_1 = \psi PF_2$ or, by 2.19, $F_1 = PF_2$.

To show the uniqueness of F_2, assume another F'', so that $F_1 = PF_2 = PF''$, or $F'PF_2 = F'PF''$. By 2.43 and 2.19, this gives $\psi F_2 = \psi F''$ and $F_2 = F''$.

2.47. *Every form F is uniquely expressible as a product of prime forms.*

By 2.42, F is contained in some prime form, say P_1, and, by 2.46, $F = P_1F_1$. By parity, $F_1 = P_2F_2, \cdots, F_{r-1} = P_rE$. Thus $F = P_1P_2 \cdots P_r$.

To show the decomposition unique, assume another decomposition, so that

$$F = P_1P_2 \cdots P_r = P_1'P_2' \cdots P_s'.$$

By 2.16 and 2.14, P_1 contains $P_1P_2 \cdots P_r$ and also $P_1'P_2' \cdots P_s'$. From this it follows, by 2.38 and 2.41, that $P_1 = P_1'$. Hence $P_2P_3 \cdots P_r = P_2'P_3' \cdots P_s'$. By parity, $P_2 = P_2'$, $P_3 = P_3'$, $\cdots$, $P_r = P_r'$ $(r \leq s)$. So, by 2.44, $P_{r+1}' \cdots P_s' = E$ and, finally,

$$P_1'P_2' \cdots P_r' \cdots P_s' = P_1'P_2' \cdots P_r'E = P_1'P_2' \cdots P_r'.$$

2.48. *If one form F_1 is contained in another form F_2, then there exists an unique form F_3 such that $F_1 = F_2F_3$.*

By hypothesis F_1 is contained in F_2, and so F_1F' is contained in F_2F'. By 2.43, F' can be chosen so that $F_1F' = \psi$, a quantity in the original domain. Thus F_1F' is contained in ψ or, by 2.18, there exists a form F_3 such that $F_1F' = \psi F_3$. From this, $F_1F_2F' = \psi F_2F_3$ or $\psi F_1 = \psi F_2F_3$. The theorem follows by 2.19.

To show F_3 unique, assume an alternative F'''. But from $F_1 = F_2F_3 = F_2F'''$ it follows, by 2.45, that $F_3 = F'''$.

This completes the restoration of the arithmetic properties of unique factorization and decomposition. Thus the ordinary laws of divisibility again hold.

In order to facilitate the translation from the present theory (*) to that

of Dedekind and of Kronecker, a dictionary of equivalent symbols and terms follows:

*	*D*	*K*
F	Ideal $\mathfrak{a}$	Form
E	Order $\mathfrak{o}$	Primitive form
P	Prime ideal $\mathfrak{p}$	Prime form
Γ	Norm of ideal $N(\mathfrak{a})$	Norm of form

3. By giving various interpretations to the symbols and relations, we obtain particular instances of the same abstract general theory. The development of section 2 made fundamental the composition of forms, as in the work of Kummer. However, this entire section could have been based on the Gaussian transformation of forms. In fact, the two ideas form two possible interpretations of the same general concept. Dedekind's Führer or Bearer furnishes another possible viewpoint.

Weber, in his Algebra, develops a theory of functionals, wherein integral associate functionals correspond to ideals; the product of two functionals to the product of two corresponding ideals; an integral number to a principal ideal; and units to the ideal or order $\mathfrak{o}$ of Dedekind.

Geometrically, an ideal may be made to correspond to a system of points on a fixed fundamental curve; and a principal ideal to the complete intersection system of the fundamental curve and another algebraic curve. Brill and Noether translate the idea of equivalence into coresiduality. Hensel and Landsberg have developed an extensive application of modular-systems to the theory of algebraic functions of curves on a surface.

ALABAMA POLYTECHNIC INSTITUTE.

Concerning the Cut Points of a Continuous Curve when the Arc Curve, AB, Contains Exactly N Independent Arcs.*

By Norman E. Rutt.

In 1925 Menger † introduced the term "curve" for those compact connected spaces which have the property that for every point of the set and every $\epsilon > 0$ there exists a neighborhood of diameter less than ϵ which contains the point and has a totally disconnected ‡ boundary. A point of such a set is said to be *regular* if it is possible to choose these neighborhoods in such a manner that its boundary is finite, while it is *of order n* if for every ϵ we can find the neighborhood of diameter less than ϵ whose boundary consists of exactly n points. In a subsequent paper he shows that in a regular curve, *i. e.*, a curse containing only regular points, there is corresponding to every point of order n a set of n arcs having the point in common and otherwise distinct.§ It should be noted that every regular curve of Menger is also a continuous curve,¶ that is, it is connected im kleinen at every point. Of course there are continuous curves which are neither regular nor even curves in the sense defined above.

In this paper is considered a plane continuous curve in which there are two points a and b which have the property that *there are in the continuous curve exactly n arcs from a to b which have no points other than a and b in common while it is impossible to find a similar set of $n+1$ arcs from a to b.* Under these hypotheses it is proved that there are exactly n points in the curve whose omission separates a from b in the curve. This theorem provides a sort of converse to the theorem of Menger mentioned. The precise converse of the theorem of this paper is an easy consequence of the theorem itself, and

* Presented to the American Mathematical Society, May 7, 1927.

† Cf. K. Menger, "Grundzüge einer Theorie der Kurven," *Mathematische Annalen*, Vol. 95 (1927), p. 277.

‡ A set of points is said to be *totally disconnected* if it contains no connected subsets other than single points.

§ See "Zur allgemeine Kurventheorie," *Fundamenta Mathematicae*, Vol. 10, (1927), p. 98.

¶ For fundamental definitions and properties of continuous curves see R. L. Moore, "Report on Continuous Curves from the Viewpoint of Analysis Situs," *Bulletin of the American Mathematical Society*, Vol. 29 (1923), pp. 289-302.

thus furnishes a proof of Menger's theorem for more general *plane continuous curves.* In the course of the argument very important use is made of the arc-curve $E(ab)$ of the continuous curve E with respect to the distinct points a and b, i. e., the set of all points $[X]$ *of* E such that for every point x of the set there is an arc of E from a to b which contains x.*

The statement of the principal theorem of this paper is due to Professor J. R. Kline, and it has been with his advice and direction that the proof has been constructed.

1. GENERAL THEOREM. *If there exists an integer n such that for two given points a and b of a continuous curve E there are arcs ax_ib $(i = 1, 2, \cdots, n)$ of E no two of which have any points except a and b in common, and such that by no means is it possible to construct $n+1$ arcs like these from a to b, then there exists a set of n distinct points $p_1, p_2, \cdots, p_n$ of E such that $E - (p_1 + p_2 + \cdots + p_n)$ is the sum of two mutually separated sets E_a and E_b, E_a containing a and E_b containing b.*

Notation and definitions. Let F be the arc curve of the point set $a+b$ with respect to E, that is, let F be the set of all arcs that can be drawn in E from a to b. It is a continuous curve every complementary domain of which has a simple closed curve as its boundary.† Let a particular set of n arcs ax_ib satisfying the hypotheses of the theorem be $M_1, M_2, \cdots, M_n$ (such a set will hereafter be referred to as *independent*) and let these be ordered in such a way that the simple closed curve formed by M_{i-1} and M_{i+1} contains in its interior (M_i), that is $M_i - (a+b)$, but not (M_j), where $j = 1, 2, \cdots, n$ and $i = 2, 3, \cdots, (n-1)$ and $j \neq i$. Let M_{ij} be the simple closed curve $M_i + M_j$, and let its interior be I_{ij}. Let F_{ij} represent $M_{ij} + F \times I_{ij}$. F_{ji} is a continuous curve;‡ the arc-curve of $(a+b)$ with respect to F_{ij} is F_{ij}.§ Let any complementary domain of F_{jj+1} whose boundary has at least one point on each of the arcs (M_j) and (M_{j+1}) be called a *cross-domain* or *c-domain* of F.

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* This notion has been introduced by W. L. Ayers in a paper which has been accepted for publication by the *Transactions of the American Mathematical Society.* Ayres proves that the arc curve $E(ab)$ is itself a continuous curve. For an abstract of this paper see the *Bulletin of the American Mathematical Society,* Vol. 33 (1927), p. 410. As this paper is as yet unpublished the numerous references to it throughout these pages are somewhat incomplete.

† W. L. Ayres, "Concerning the Arc-curves and Basic Sets of a Continuous Curve"; see remarks above in introduction.

‡ H. M. Gehman, "Concerning the Subsets of a Plane Continuous Curve," *Annals of Mathematics,* Vol. 27, 1925, No. 1, p. 34.

§ W. L. Ayres, "Concerning the Arc-curves and Basic Sets of a Continuous Curve."

In what follows ordinarily c-domains will be represented by small Greek letters γ, δ, their boundaries by the corresponding capitals Γ, Δ.

I(a) THEOREM. *At least one of the complementary domains of F_{ii+1} is a cross-domain of F.*

Proof: Consider the set $F_{12} - M_{12}$. If this is vacuous the theorem is obvious as then I_{12} is a c-domain. Since F_{12} and M_{12} are both continuous curves and $M_{12} \subset F_{12}$, if $F_{12} - M_{12}$ is not vacuous, it consists of a certain number of maximal connected parts. Suppose that one of these, say K, has limit points both upon (M_1) and upon (M_2). If both a and b are also limits of K there must in K be an arc from a to b.* Either this is an $n+1st$ arc from a to b contradicting hypothesis, or it is one of the arcs M_j, $j > 2$, contrary to the convention used in ordering $M_1, \cdots, M_n$. Suppose then that a is not a limit point of K. As the limit points of K form a closed set with representative points both upon (M_1) and upon (M_2) there is a first k_1 on the one and a first k_2 on the other. Now $M_{12} - (k_1 + k_2)$ is disconnected into $(k_1 a k_2)$ and $(k_1 b k_2)$. The arcs $(k_1 a k_2)$ and $(k_1 b k_2)$ will then occur in the same maximal connected subset of $F_{12} - (k_1 + k_2)$ only if some maximal connected subset of $F_{12} - M_{12}$ has limit points on both $(k_1 a k_2)$ and $(k_1 b k_2)$. Suppose one of these, say L, has one, l_a, in $(k_1 a k_2)$ and one, l_b, in $(k_1 b k_2)$. Draw an arc $(k_1 x k_2)$ in K and one $(l_a y l_b)$ in L, as these must intersect interior to I_{12} and thus K be identically L, a contradiction arises which shows that $F_{12} - (k_1 + k_2)$ is disconnected. But then k_1 and k_2 are on the boundary of a certain bounded complementary domain of F_{12}.†

Suppose then that no maximal connected subset of $F_{12} - M_{12}$ has a limit point on (M_1) and another on (M_2). As before $F_{12} - (a + b)$ is disconnected and so a and b lie on the boundary of some bounded complementary domain D of F_{12}. The boundary of this domain is a subset of F_{12} and is a simple closed curve which is divided into two arcs (ayb) and (azb) by the points a and b. As neither of these arcs has any points exterior to M_{12} it is clearly impossible for them both to have points on either (M_1) or (M_2), or for one to have points on both (M_1) and (M_2). But if (ayb) has no point on (M_1) it must have at least one on (M_2) as otherwise it is an $(n+1)st$ arc from a to b in F. Then similarly (azb) has a point on (M_1). Thus as one of the two arcs has a point on (M_1) and the other one on (M_2) the domain

* R. L. Wilder, "Concerning Continuous Curves," *Fundamenta Mathematicae*, Vol. 7, p. 342.

† G. T. Whyburn, Abstract of an unpublished paper in the *Bulletin of the American Mathematical Society*, Vol. 33, p. 388, No. 10.

which they bound is a cross-domain of F. The same result of course follows when (ayb) has a point on (M_1).

The theorem is thus true for F_{12}. But as the argument used in the proof involved only F_{12} itself it evidently applies immediately to each set F_{jj+1} $(j = 2, 3, \cdots, n-1)$.

Notation and definitions. Let δ be any c-domain of F_{jj+1}, that is of F in I_{jj+1}. Its boundary Δ has a first point a_j on M_j and also a last b_j, likewise it has a first and last a_{j+1} and b_{j+1} on M_{j+1}. It may happen, (1) that $a_j \equiv a \equiv a_{j+1}$, (2) that $b_j \equiv b \equiv b_{j+1}$, (3) that $a_j \equiv a \equiv a_{j+1}$ and $b_j \equiv b \equiv b_{j+1}$. In these cases it will be convenient still to speak of these coincident pairs as if they were distinct points.

The symbols a_jxa_{j+1} and b_jyb_{j+1} each represent two arcs or points of Δ; of these one from the first pair and one from the second have at most end points in common, and these two will be called the *a- and b-arcs of* δ, Δ, even although they may reduce to points. If γ and δ are c-domains of F_{jj+1} and γ precedes δ these two will be said to be *adjacent* when the b-arc of Γ, the a-arc of Δ, and the subarcs of M_j and M_{j+1} which their ends intercept either fail to form a simple closed curve or form one whose interior contains no c-domain of F.

I(b) THEOREM. *If γ and δ are adjacent c-domains of F_{jj+1}, γ preceding δ, then from a point of the b-arc of Γ in I_{jj+1} to a point of the a-arc of Δ in I_{jj+1} there is an arc of $F_{jj+1} - M_{jj+1}$ having only an end point in common with either Γ or Δ.*

Proof: The argument will be given only for F_{12}. Let the *a- and b-arcs* in question be b_1yb_2 and a_1za_2. If $(b_1yb_2) \times (a_1za_2) \neq 0$ any point of the intersection will be considered to satisfy the theorem. If this product is zero the two arcs involved in it form with the arcs of (M_1) and (M_2) which their ends intercept a simple closed curve C with interior I. If I contains no constituent of F with limits on both (b_1yb_2) and (a_1za_2) by an argument like that in I(a) I contains a c-domain of F. Hence the theorem. Clearly the theorem is similarly true in the special cases $\gamma + \Gamma = a$, and $\delta + \Delta = b$.

I(c) THEOREM. *If the c-domains of F_{jj+1} form an infinite set this set must have at least one limit point not already a limit point of some finite subset of the set, and only a or b can be such a limit point.*

Proof: That one such limit must exist is evident from the fact that the domains in question form a bounded infinite set. Suppose c is one distinct from a and b. At least one of the distances c to (M_j), c to (M_{j+1}) is not zero; suppose the first is ϵ. Let c be the center of a circle of radius $\epsilon/2$. Within

this circle an infinite number of the c-domains in question must have points, and as each of these must also have a boundary point on (M_j) each c-domain of this infinite number must have diameter greater than $\epsilon/2$, an evident contradiction.

It may also be noted that if a is one of the limits of the c-domains of F_{jj+1} it can not be a boundary point of any c-domain of F_{jj+1}, for if it is a point of Δ and if p_j and p_{j+1} are points of Δ on (M_j) and (M_{j+1}), (p_jxp_{j+1}) being an arc of δ, p_jyb one on M_j and $p_{j+1}zb$ one of M_{j+1}, then a is exterior to the simple closed curve $p_jxp_{j+1}zbyp_j$ while all but δ of the set of c-domains of F_{jj+1} of which a is a set limit are interior to this curve. It is clear that the theorem may be stated equally truly for the boundaries of the c-domains of F_{jj+1}.

Notations and definitions. Owing to I(c) the c-domains of F_{jj+1} may be represented as $\cdots, \delta_{-m}, \cdots, \delta_{-2}, \delta_{-1}, \delta_0, \delta_1, \delta_2, \cdots, \delta_m, \cdots, \delta_0$ having been

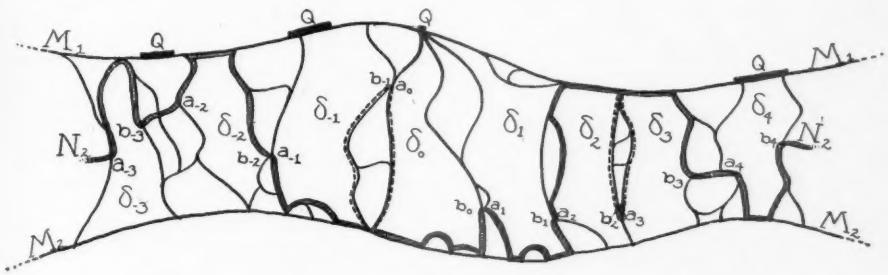

FIG. 1.

The heavy line is N_2. From the sum have been omitted the dotted loups.

selected at random. The i c-domains $\gamma_{jj+1}, \cdots, \gamma_{j+i-1j+i}$ will be said to form a *chain of* F_{jj+1} from M_j to M_{j+i} if (1) one and only one of them is a complementary domain of each of the curves $F_{jj+1}, \cdots, F_{j+i-1j+i}$; and (2) Γ_{q-1q} has at least one point of (M_q) in common with Γ_{qq+1} $(q=j+1, \cdots, j+i-1)$. $\Gamma_{jj+1}, \cdots, \Gamma_{j+i-1j+i}$ will be referred to as *links* of the chain, and the points of Γ_{jj+1} on (M_j) and of $\Gamma_{j+i-1j+i}$ on (M_{j+i}) as *end points* of the chain.

I(d) THEOREM. *If Q is any subset of (M_j) there are in F_{jj+i}, i independent arcs $N_{j+1}, N_{j+2}, \cdots, N_{j+i}$ from a to b, F_{t-1t} containing N_t $(t=j+1, \cdots, j+i)$; where $N_{j+1} \times Q = 0$, and $N_t \times (M_t)$ is a subset of the end points on M_t of the chains of F_{jt} which have at least one opposite end point among the points of Q.*

For the sake of simplifying the subscripts this theorem without loss of generality will be proved for F_{11+i}. The proof will be by induction. During the first part refer to fig. 1.

Proof: Let the set of all the c-domains of F in F_{12} be represented by $\sum_i \delta_i = \cdots + \delta_{-s} + \cdots + \delta_0 + \cdots + \delta_t + \cdots$ and the set of their boundaries by $\sum_i \Delta_i = \cdots + \Delta_{-s} + \cdots + \Delta_0 + \cdots + \Delta_t + \cdots$. Let $\{\Delta\}$ be the subset of $\sum_i \Delta_i$ consisting of all and only those for which $\Delta_i \times Q \neq 0$. If a is not a limit of the set $\sum_i \delta_i$ it will be convenient to consider it the first of the set $\sum_i \Delta_i$, under similar circumstances to consider b the last; under this convention $\sum_i \Delta_i$ will necessarily contain three or more members. If Δ_{i-1} and Δ_i are adjacent elements of $\sum_i \Delta_i$ let $b_{j-1}a_j$ be the arc of I(b) (of course possibly $b_{j-1}a_j$ represents a point, even maybe a or b) for every j, and let similarly a_jb_j be that arc of Δ_j for which $a_jb_j \times M_1 = 0$ if $\Delta_j \times Q \neq 0$ otherwise for which $a_jb_j \times M_1 \neq 0$. Now the sum $a_{j-1}b_{j-1} + b_{j-1}a_j + a_jb_j$ is either an arc from a_{j-1} to b_j or fails to be one only because $(a_{j-1}b_{j-1}) \times (a_jb_j) \neq 0$ In the latter case, however, a proper subset of $a_{j-1}b_{j-1} + a_jb_j$ is an arc from a_{j-1} to b_j and the sum above will be then supposed to represent this arc; the same meaning is to be assigned to the more extensive sums below. Thus $b_{-r-1}a_{-r-1} + a_{-r-1}b_{-r} + b_{-r}a_{-r+1} + \cdots + b_{-2}a_{-1} + a_{-1}b_{-1} + b_{-1}a_0 + a_0b_0 + b_0a_1 + a_1b_1 + b_1a_2 + \cdots + b_{s-1}a_s + a_sb_s + b_sa_{s+1}$ is a simple continuous arc from b_{-r-1} to a_{s+1} say N_2 for which $N_2 \times Q = 0$ and $N_2 \times (M_2) = \{\Delta\} \times (M_2)$ by reason of the choice of a_jb_j. If a finite r and s exist for which $\Delta_{-r-1} \equiv a$ and $\Delta_{s+1} \equiv b$ the first step in the induction is now established. If one or both subscripts fail to exist continuing the process indefinitely toward a or b gives rise to a set which becomes an arc upon the addition of its limits which by I(c) are at most a or b. In all cases then is the first step complete.

Assuming now that in F_{1k} from a to b there are $k-1$ independent arcs $N_2, \cdots, N_k$ where $1 < k < i$, $N_2 \times Q = 0$, and $N_k \times (M_k)$ is a subset of the ends on (M_k) of those chains of F_{1k} among the ends on (M_1) of each of which there is at least one point of Q, the same thing for one greater subscript is to be proved. Let $\sum_i \gamma_i$ represent the set of c-domains of F_{kk+1}, and let $Q_k = N_k \times (M_k)$. By the first part of the proof there is in K_{kk+1} from a to b an arc N_{k+1} for which $N_{k+1} \times Q_k = 0$, hence an arc independent of $N_2, \cdots, N_k$, and for which $N_{k+1} \times (M_{k+1})$ is a subset of $\{\Gamma_{kk+1}\} \times (M_{k+1})$ where $\{\Gamma_{kk+1}\}$ represents the subset of all the boundaries Γ_i of domains of $\sum_i \gamma_i$ for which $\Gamma_i \times Q_k \neq 0$. But if $\Gamma_i \times Q_k \neq 0$ then $\Gamma_i \times (M_k)$ includes an end of some chain from (M_1) to (M_k) and γ_i is a last domain of a chain from (M_1) to (M_{k+1}) so $\Gamma_i \times (M_{k+1})$ consists of ends on (M_{k+1}) of chains of F_{1k+1} whose ends on (M_1) include at least one point of Q for each chain, since the points of Q_k are the ends of similar chains of F_{1k}. It is to be noted that this construction makes $N_t \times (M_t)$ not a proper subset of the end points on M_t of the chains specified. The induction, however, is complete.

I(e) COROLLARY. *If* $Q_{t+1} = (M_{t+1}) \times \{\Gamma_{tt+1}\} (t = 1, 2, \cdots, i-1)$ *then* $Q_{t+1} = (M_{t+1}) \times N_{t+1}$ *and if* $\{\bar{\Gamma}_{tt+1}\} = \sum_i \Gamma_i - \{\Gamma_{tt+1}\}$ *then*

$$N_{t+1} \times (M_t) = \{\bar{\Gamma}_{tt+1}\} \times (M_t) \subset (M_t) - Q_t.$$

The construction used above demonstrates this.

I(f) COROLLARY. *In* F_{1n} *there is at least one chain from* (M_1) *to* (M_n).

For suppose not and apply I(d) to F_{1n} with $Q = (M_1)$. Then in F_{1n} there are $n + 1$ independent arcs from a to b, namely M_1, N_2, N_3, $\cdots$, N_{n-1}, N_n, and M_n as clearly $(M_1) \times N_2 = 0$ and $N_n \times (M_n) = 0$.

Remarks: The general theorem is now proved, except for the final steps in § 5, in case both a and b are on the simple closed curve which forms the outer boundary of F; for in this case M_1 and M_n may always be chosen as the two arcs into which a and b divide this curve and the statements I(a) to I(f) inclusive still hold. The unbounded complementary domain of F together with any chain of F_{1n} existing by I(f) constitute the set of n domains employed in § 5. In §§ 2, 3, 4 it will accordingly be assumed that either a or b is not on the outer boundary of F. By transformations of simple sorts some of the cases in which this is true can be reduced to the case already treated, but as these are special it seems not worth while to treat them separately.

The points of F_{1n} which are either end points of chains of F_{1n} or are points common to two links of some such chain are of special importance and will be called *invariant*. As already mentioned, if $F_{1n} \equiv F$ the argument of § 5 applies without any further discussion with the consequence that the set of invariant points of F_{1n} is a subset of every set of n independent arcs of F_{1n} from a to b.

2. An analysis of the non-vacuous set $F - F_{1n}$ must now be made. Let it be G; it consists of a set of maximal connected sets of which at most a finite number are of diameter greater than any assigned positive number ϵ,* and each of which has at least two limit points in M_{1n}.† Let $G = H + I + J + K$ where, H consists of those constituents of G all whose boundary points are in either (M_1) or (M_n) but not both, and none of which contains an arc which with an arc of (M_1) (or of (M_n)) forms a simple closed curve having I_{1n} in its interior (the sets making up H are at most countably infinite); J consists of sets at most finite in number differing in no respect from those in H except that a simple closed curve as described in connection

* W. L. Ayres, "Concerning Continuous Curves and Correspondences," *Annals of Mathematics*, Second Series, Vol. 28, No. 3, 1927, July, p. 396.

† W. L. Ayres, "Concerning Arc Curves and Basic Sets of a Continuous Curve."

with H now exists containing I_{1n} in its interior; I is made up of all those constituents of G among whose boundary points are a or b but obviously not both, it consists of subsets A, all whose maximal connected parts have a as limit, and B, similarly defined; K is composed of all those constituents of G not already included in I and having at least one limit on each of (M_1) and (M_n).

II(a) THEOREM. *The set $F_{1n} + H$ possesses all the properties of F_{1n} that were derived in section* 1.

Let any maximal connected part of H be H_i. Now $\overline{H}_i$, the set plus its limit points, is a continuous curve.* As F_{1n} is a continuous curve and as for every i, $\overline{H}_i \times F_{1n} \neq 0$, $F_{1n} + \sum_i \overline{H}_i$ is a continuous curve.† But since $\sum_i \overline{H}_i - H \subset F_{1n}$, $F_{1n} + \sum_i \overline{H} = F_{1n} + H$ and so $F_{1n} + H$ is also a continuous curve. The arc curve of $a + b$ with respect to $F_{1n} + H$ is $F_{1n} + H$, so the outer boundary of $F_{1n} + H$ is a simple closed curve. This simple closed curve must have a point on each of (M_1) and (M_n) as otherwise H must have contained constituents properly belonging to J or K. Let now $H + M_{1n}$ be inverted with center of inversion a point of I_{1n}, and M_{1n} as invariant simple closed curve. The unbounded complementary domain of $F_{1n} + H$ becomes a bounded cross-domain of $\Pi(M_{1n} + H)$ and as H had no constituent with limit points on both (M_1) and (M_n), $\Pi(M_{1n} + H) - M_{1n}$ has no constituent with limit points in both (M_1) and (M_n). Thus as in I(a) a and b are on the boundary of a bounded complementary domain of $\Pi(M_{1n} + H)$ which is also a c-domain of it. When a c-domain of this type exists it is clearly the only possible c-domain and so must be the inverse of the unbounded complementary domain of $F_{1n} + H$ which accordingly has a boundary containing a and b. The two arcs into which a and b divide this boundary may be taken as M_1 and M_n. The theorem is thus proved. Hereafter F will represent $F_{1n} + H$ so that $G = I + J + K$.

If $A \neq 0$, it has last points i_{1a} and i_{na} on M_1 and M_n respectively perhaps not both different from a but clearly both different from b. With regard to B there are similar points.

II(b). *A contains arcs $(ax i_{1a})$, $(ax i_{na})$ and B contains similar arcs.*

For a is a limit of every constituent of A and if i_{1a} were not a limit of some particular one there would be a contradiction.

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* See second footnote referred to in section 1 and in the theorem there used read "exterior" as well as "interior."

† H. M. Gehman, "Some Conditions under which a Continuum is a Continuous Curve," *Annals of Mathematics*, Second Series, Vol. 27, p. 381.

II(c). *All the limits of J occur on an arc of (M_1) [or (M_n)] which except perhaps at its ends contain no limits of I or K.*

II(d). *The only limit points, not already limits of a particular constituent, which K can have are a and b.*

The set K needs to be discussed in detail. Obviously the limits of any constituent of it on (M_1) or (M_n) occupy an interval whose interior points are none of them limits of any other constituent. In every K_j draw an arc from a limit on (M_1) to one on (M_n). This arc forms with an arc of M_{1n} going through a a simple closed curve whose interior will be called I_j. It may happen for some arcs in K_j that I_j contains I_{1n} and for some not; clearly only one constitutent of K_j can have this property, if such a constituent exists it will be represented by K_0. For each of the remaining constituents K_j of K either I_j never contains I_{1n} in which case j will be considered negative or I_j always contains I_{1n} in which case j will be positive. If i and j are negative and $I_i \subset I_j$ then let i and j be integers such that $i < j$, whereas if i and j are positive and $I_i \supset I_j$ let i and j be integers and $i < j$. In view of II(d) and obvious facts, this representation is possible and may be taken as $\cdots$, $K_{-l}, \cdots, K_{-2}, K_{-1}, K_0, K_1, K_2, \cdots, K_l, \cdots$, where of course K_0 may be vacuous. Precedence among the constituents of K will hereafter be determined by referring to their subscripts.

The constituent K_j has on (M_1) and (M_n) first limit points a_{1j} and a_{nj} and last limit points b_{1j} and b_{nj}. On (M_1) the interval $b_{1j}a_{1j+1}$ will be called C_j, on (M_n) the interval $b_{nj}a_{nj+1}$ will be called D_j, and C_j, D_j will be referred to as a pair of *corresponding intervals*. If $K_0 = 0$ C_0 and C_{-1} will represent the same interval, D_0 and D_{-1} likewise. If $A \neq 0$ there is a first constituent of K say K_{-l} and a first pair of corresponding intervals $C_{-l-1} = i_{1a}a_{1,-l}$, and $D_{-l-1} = i_{na}a_{n,-l}$, one but not both of which may begin with a. Even when A is vacuous there may of course be a first K_{-l} with both C_{-l-1} and D_{-l-1} beginning at a. Relations between K and B of course present no features not possible between K and A. Obviously C_j and D_j may reduce to points but will nevertheless be referred to as intervals. When a (or b) is end of an interval the interval will be assumed to lack this point.

II(e). *Any pair of corresponding intervals is on the boundary of a common complementary domain of F.*

Assuming K and J vacuous the theorem is obvious as then $i_{1a}i_{1b} = C_0$ and $i_{na}i_{nb} = D_0$ are the only corresponding intervals and are on the outer boundary of F. If $J = 0$ but $K \neq 0$ invert $F - F \times I_{1n}$ from any point of I_{1n} leaving M_{1n} invariant and getting a set $\bar{F}$. By the same argument as that

of the first case of I(a) any point c of C_j and any point d of D_j are on the boundary of a common bounded complementary domain of $\bar{F}$, and any other point c' of C_j can fail to be on the same boundary only if a constituent of $\bar{F} - M_{1n}$ has a limit point interior to C_j, which is ridiculous; similarly any point d' of D is on the boundary of this same domain. Hence C_j and D_j are on the boundary of a common complementary domain of F. If the arc of II(c) occurs on one of a pair of corresponding intervals as it may if $J \neq 0$ it can occur only on C_0 or D_0 and on one of these only if $K_0 = 0$. In this case it again separates $C_0 = C_{-1}$ (or $D_0 = D_{-1}$) into two distinct sets both of which are easily proved to be on the boundary of the same domain as $D_0 = D_{-1}$ (or $C_0 = C_{-1}$).

The relation of the chains of F_{1n} to the pairs of corresponding intervals is of critical importance. Let C and D be any corresponding pair. If at least one chain of F_{1n} has ends on both C and D, these two are said to be *joined*. If no chain has ends on C (or D) then C (or D) is said to be *free*. If the only chains having ends on C (or D) have opposite ends all preceding D (or C) then C (or D) is said to be *a-joined*; if all following, C (or D) is said to be *b-joined*; if both preceding and following, C (or D) is said to be *ab-joined*. The following is a complete list of the possible relations of C and D to the chains of F_{1n}:

1 C free, D free
2 C free, D *a*-joined
3 C free, D *b*-joined
4 C free, D *ab*-joined
5 C *a*-joined, D free
6 C *b*-joined, D free
7 C *ab*-joined, D free
8 C *a*-joined, D *b*-joined
9 C *b*-joined, D *a*-joined
10 C *a*-joined, D *a*-joined
11 C *a*-joined, D *ab*-joined
12 C *b*-joined, D *b*-joined
13 C *b*-joined, D *ab*-joined
14 C *ab*-joined, D *a*-joined
15 C *ab*-joined, D *b*-joined
16 C *ab*-joined, D *ab*-joined
17 C and D joined.

The object of the next two sections is to show that the fundamental assumptions of the theorem imply that at least one pair of corresponding intervals is joined (relation 17). Section 3 is devoted to some preliminary considerations.

3. In any chain of F_{1n} an arc may be drawn from an end of the chain on (M_1) to one on (M_n) by way of domains of the chain with the exception of precisely n points one on each of the arcs (M_j) $(j = 1, \cdots, n)$ necessarily invariant. Such an arc will be called an *arc of the chain*. Two important arcs may also be drawn on the boundaries of domains of the chain.

Let an arc be drawn from the last end of the chain on M_1 to the last end on M_n so as to contain the b-arc of every link of the chain, this will be called the *b-boundary arc* of the chain. It may reduce to point b. The *a-boundary arc* is similarly defined. Special cases of no significance arise if more than one domain of the chain has a or b as limit.

III(a) THEOREM. *If on M_1 there is the order $a\,c_1d_1b$ and on M_n a d_nc_nb and c_1, c_n are ends of a chain $\gamma_1, \cdots, \gamma_{n-1}$, d_1, d_n of a chain $\delta_1, \cdots, \delta_{n-1}$; then c_1, d_n are ends of a chain, also c_n, d_1.*

Proof: Construct an arc c_1c_n of $\gamma_1, \cdots, \gamma_{n-1}$ and one d_1d_n of $\delta_1, \cdots, \delta_{n-1}$. Owing to the given order of their end points these two arcs intersect in I_{1n}. Counting from c_1 let p be the first common point. If p is interior to some curve M_{jj+1}, $(j = 1, 2, \cdots, (n-1))$ then γ_j and δ_j coincide as p is interior to both; accordingly $\gamma_1, \cdots, \gamma_{j-1}, \gamma_j \equiv \delta_j, \delta_{j+1}, \cdots, \delta_{n-1}$ and $\delta_1, \cdots, \delta_{j-1}, \delta_j \equiv \gamma_j, \gamma_{j+1}, \cdots, \gamma_{n-1}$ are the chains fulfilling the requirements. If p is on M_j then p is in each of the sets Δ_{j-1}, Δ_j, Γ_{j-1}, and Γ_j. But then $\gamma_1, \cdots, \gamma_{j-1}$, $\delta_j, \cdots, \delta_{n-1}$ and $\delta_1, \cdots, \delta_{j-1}, \gamma_j, \cdots, \gamma_{n-1}$ answer the theorem.

This theorem shows that cases 10 to 16, inclusive, at the end of section 2 all reduce to 17 and from now on will be omitted. A common feature of cases 1 to 7 inclusive is the occurrence of at least one free interval. Let us consider the situation.

Any free interval must be a subset of some maximum free segment at least one end of which (but not necessarily both, as one may be a or b) is an end of a chain. Let (l_1f_1) be a maximum free segment with $f_1 = b$. Let γ_1 be the last c-domain of F_{1n} which is a link of a chain having l_1 on Γ_1. If Γ_1 contains b complete the chain $\gamma_1, \cdots, \gamma_{n-1}$ in any convenient way. If not let γ_2 be the last c-domain of F_{23} which is part of a chain from $al_1 \subset M_1$ to (M_n) for which $\Gamma_1 \times \Gamma_2 \neq 0$. Either Γ_2 contains b in which case complete the chain $\gamma_1, \gamma_2, \cdots, \gamma_{n-1}$ in any convenient way, or it does not. In this case let γ_3 be the last c-domain of F_{34} which is part of a chain from al_1 to (M_n) with $\Gamma_2 \times \Gamma_3 \neq 0$. If b is on Γ_3 complete the chain at random, if not repeat the process above, and so on until a link of a chain is thus found with b on its boundary, or a complete chain consisting of last possible links is found from l_1 to an extreme point l_n on (M_n). Draw the b-boundary arc of the chain above until it reaches b (call it then l_1lb) or until it reaches (M_n) (call it then l_1ll_n).

If l_1 is a, then a similar construction working toward a results either in an arc f_1fa or an arc f_1ff_n. If $a \neq l_1$ and $f_1 \neq b$ then either l_1 and f_1 are on the boundary of some common c-domain of F_{12} or there is from l_1 an arc which is b-boundary arc of a chain consisting of last possible domains to an

extreme point l_n on (M_n), and from f_1 to f_n (an extreme point of (M_n) toward a) an arc which is a-boundary arc of a chain of first possible domains.

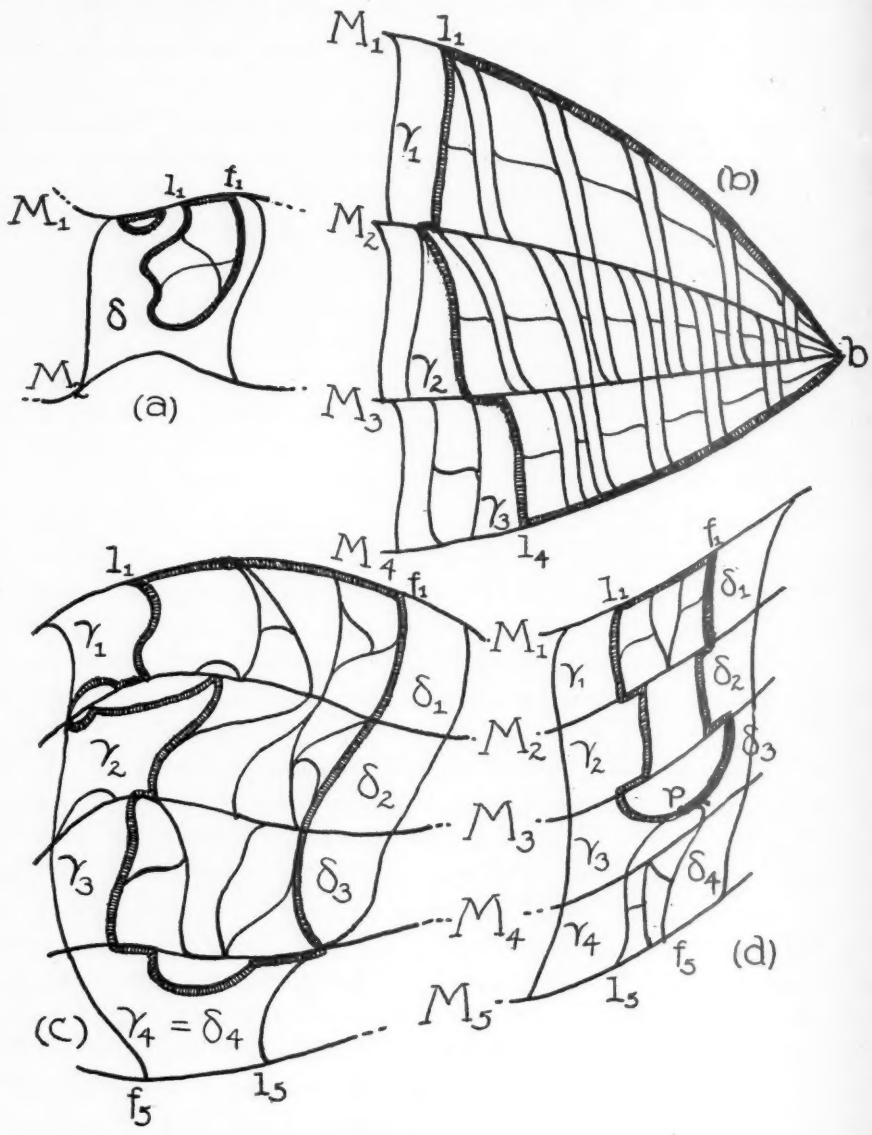

FIG. 2.

Pockets are enclosed by heavy line.

The arcs l_1l_n and f_1f_n may not intersect but if they do proceeding from l_1 call the first common point p and in the sum of the arcs draw the arc l_1pf_1, where possibly $l_n = p = f_n$.

Now by means of the minor * arcs of M_{1n} complete the following simple closed curves; (1) if l_1 and f_1 are on the same Δ join them along the minor arc * of Δ and the minor arc of M_{1n}, fig. 2(a); (2) if there is only one chain and arc, l_1lbl_1, $l_1ll_nbl_1$, f_1faf_1, $f_1ff_naf_1$, fig. 2(b); (3) if there are two arcs intersecting, $l_1pf_1l_1$, fig. 2(c); (4) if there are two arcs not intersecting, $l_1l_nf_nf_1l_1$. These curves and the subsets of F_{1n} which they enclose will be called *pocket subsets* or *pockets* of F_{1n} determined by (l_1f_1). Clearly maximum free segments on M_n as say (l_nf_n) will determine similar sets in connection with it. Evidently the boundary of any pocket of F_{1n} with respect to F_{1n} consists of a finite number of invariant points unless the point p happens to exist and be not invairant, fig. 2(d).

III(b) THEOREM. *In the pocket subset of* (l_1f_1) (or (l_nf_n)) *the arcs* $M_1, M_2, \cdots$, (*or* $M_n, M_{n-1}, \cdots$) *may be diverted in such a way that* $M_1 \times (l_1f_1) = 0$ (*or* $M_n \times (l_nf_n) = 0$) *and yet the set remains independent.*

Proof: The theorem is obvious in case (1) above as there M_1 may merely follow the minor arc of Δ, $M_2, \cdots, M_n$ being undisturbed. The other cases will be handled by I(d). Let $Q = (l_1f_1)$, and construct the arcs N_2, N_3, $N_4, \cdots$. Suppose $l_2, l_3, \cdots$, are the last points common to successive links of $\gamma_1, \cdots, \gamma_{n-1}$ the chain described in connection with the pocket above, and $f_2, f_3, \cdots$, the first points common to the successive links of $\delta_1, \cdots, \delta_{n-1}$ the chain of first links also used. As seen in the discussion of pocket subsets above one or other of these chains may fail to exist so that the corresponding set of points just mentioned reduces either to a or to b; but as this circumstance has the effect of simplifying matters special treatment of such a case is not necessary. Now if not vacuous $Q_2 = N_2 \times M_2 = M_2 \times \{\Delta_{12}\}$ † and so, l_2f_2 being an arc of M_2, $Q_2 \subset l_2f_2 - l_2f_2 \times (\Gamma_2 + \Delta_2)$, as otherwise, referring to III(a) and the supposition that (l_1f_1) is free, a contradiction arises. Therefore evidently l_2 precedes Q_2, and Q_2 precedes f_2. Likewise if $l_3f_3 \subset M_3$ and $Q_3 \neq 0$, then $Q_3 \subset l_3f_3 - l_3f_3 \times (\Gamma_3 + \Delta_3)$, and so on. But by supposition (l_1f_1) is free and so $Q_n = 0$, hence some last Q, say $Q_k (k < n)$, is non-vacuous. Here also if arc $l_kf_k \subset M_k$, then $Q_k \subset l_kf_k - l_kf_k \times (\Gamma_k + \Delta_k)$, evidently with l_k preceding Q_k and Q_k preceding f_k. As the boundary of the

* By minor arc of M_{1n} will here be meant only that which does not contain both a and b; by minor arc of Δ is meant that arc of it which with the minor arc of M_{1n} forms a simple closed curve not containing δ. These definitions can cause no confusion in this connection as a, l_1, f_1, b are all on M_1.

† The symbol $\{\Delta_{12}\}$ has a meaning similar to $\{\Gamma_{kk+1}\}$ in the proof of I(d).

pocket with respect to F_{1n} is a subset of $\sum_{i=1}^{n} l_i + \sum_{i=1}^{n} f_i + p$ and as by I(e) for $2 \leqq j \leqq k$ $l_{j-1} + f_{j-1} \subset N_j \times M_{j-1}$ it is now clear that between l_{j-1} and f_{j-1} N_j is interior to the pocket. Moreover, since $Q_{k+1} = 0$, N_{k+1} skirts all the c-domains of F_{kk+1} by way of M_k, and thus clearly does not leave the pocket between l_k and f_k. Between l_i and f_i let M_i coincide with N_{i+1} and be unchanged in position elsewhere $(1 \leq i \leq k)$. The arc M_1 thus avoids $(l_1 f_1)$; $M_2, \cdots, M_k$ remain independent of each other, are unchanged in position outside of the pocket, and are independent of $M_{k+1}, \cdots, M_n$ which have remained unaltered. It is to be noticed that M_k may possibly now pass through the perhaps not invariant point p, fig. 2(d).

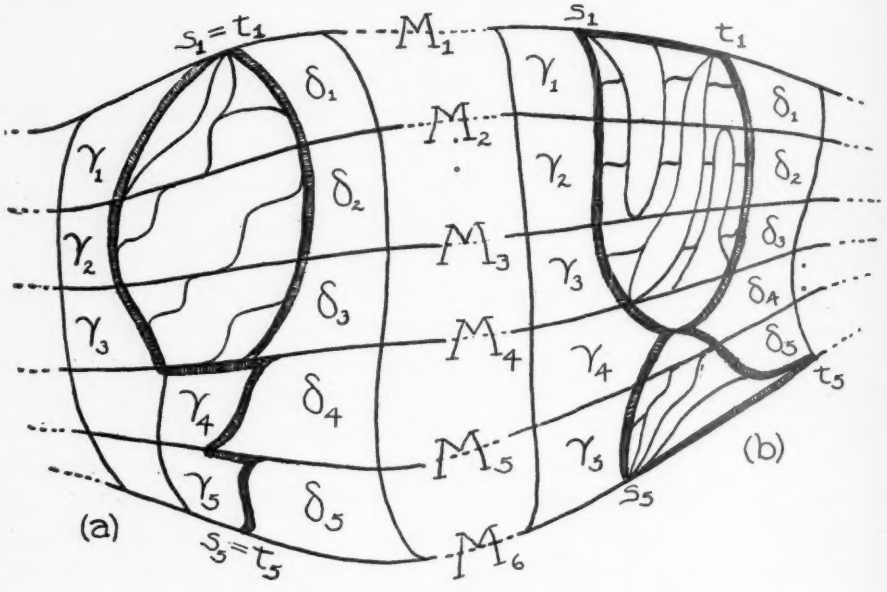

FIG. 3.
Switches $/s_n t_1/$ in heavy outline.

Cases 8 and 9 when they occur imply a condition of different significance. This will now be discussed.

Let C and D be a pair of corresponding intervals of type 8. There are on C end points of one or more chains of F_{1n} but all the opposite ends of these precede D on (M_n), the last such point preceding being say s_n. On the other hand there is on (M_1) at least one end of a chain whose ends on (M_n) follow s_n for D is not free. Let t_1 be the first point of (M_1) which is an end of a chain among whose opposite ends are points following s_n. That is t_1 is

the first end of some chain $\delta_1, \cdots, \delta_{n-1}$ with a-boundary arc $t_1t_2 \cdots t_{n-1}t_n$, t_n not preceding s_n but perhaps coinciding with it, whose links are the first ones possible satisfying the other requirements.

Two possibilities arise with regard to t_1. (1) Perhaps t_1 is a last but not only end of a chain of last domains $\gamma_1, \cdots, \gamma_{n-1}$ ($\gamma_1 \neq \delta_1$ evidently) with b-boundary arc $s_1s_2 \cdots s_{n-1}s_n$ ($t_1 = s_1$).* Here s_1 is not on C but other ends of this chain may be. This chain may have ends on (M_n) preceding s_n but none following. (See fig. 3(a)). (2) Otherwise t_1 is an only end or is not an end of chains whose first links precede δ_1 and whose last opposite end is s_n. Here as t_1 is not on C it must be preceded by some last end s_1 of a chain of last links $\gamma_1, \cdots, \gamma_{n-1}$ and b-boundary arc $s_1s_2, \cdots, s_{n-1}s_n (s_1 \neq t_1)$, fig. 3(b). The segment (s_1t_1) is free. The supposition that t_1 is a limit of preceding end points of chains reduces easily to (1).

With regard to s_n there are also two possibilities as follows: (1) $\Delta_{n-1} \times s_nb \supset \Delta_{n-1} \times M_n \supset s_n$ and $\neq s_n$ where s_nb is an arc of M_n; and (2) s_n precedes $\Delta_{n-1} \times M_n$ hence (s_nt_n) is free, t_n being first point of $\Delta_{n-1} \times M_n$. The chains γ and δ can neither cross nor have a common link. The subset of F composed of s_1s_n (b-boundary arc of γ), s_nt_n on M_n, t_nt_1 (a-boundary arc of δ), t_1s_1 on M_1, and the subset of F_{1n}, if any, which is included between them will be called a *switch subset* or *switch* of F_{1n}. The notation for it will be $/s_nt_1/$, s_1 and t_n being omitted altogether as of less importance even when distinct from s_n and t_1. The bar / indicates that C is a-joined, D b-joined, and makes the property of III(c) below easier to remember.

The discussion above was referred not only to the corresponding pair $C\,D$ but also to M_1 and point a. A similar discussion for $C\,D$ referring to point b and M_n determines a similar set but not necessarily the same one. If, however, the sets are not identical a particular pair of intervals (no longer corresponding) may be chosen instead of $C\,D$ which will when discussed from M_n and b give the identical set $/s_nt_1/$. Similarly 9 on $C\,D$ either from a and M_n or b and M_1 may be used to determine switches, the same or not, for which the symbol will be $\diagdown s_1t_n \diagdown$, as above s_i preceding $t_i (i = 1, n)$.

The most important property of the switch is the following:

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* Although s_n itself is a point of the b-boundary arc of a chain with ends on C, $s_1s_2 \cdots s_{n-1}s_n$ is not necessarily such an arc; the chain determining it may have all its ends on (M_1) following C. It is a convenience however to introduce the symbol s_n at once, before indicating which of the b-boundary arcs of which it may be an end is the most important.

III(c) THEOREM. *In the Switch $/s_nt_1/$ the arcs $M_2, M_3, \cdots, M_n$ can be shifted from the points $s_2, s_3, \cdots, s_n$ to the points $t_1, t_2, \cdots, t_{n-1}$ without losing their independence, and in such a way that M_2 reaches M_1 first at t_1 and M_n leaves its original position for good at s_n. In the switch $\backslash s_1t_n\backslash$ the arcs $M_1, M_2, \cdots, M_{n-1}$ can be shifted from $s_1, s_2, \cdots, s_{n-1}$ to $t_2, t_3, \cdots, t_n$ without losing their independence, and in such a way that M_1 leaves its original position for good at s_1 and M_{n-1} reaches M_n at t_n and not before.*

Only one of these statements will be proved as the procedure for proving the other is similar.

Proof: Refer to I(d) and I(e), and to prove the first let Q be the arc (at_1) of M_1. Construct the arcs $N_2, \cdots, N_n$. For either supposition concerning t_1 clearly the γ chain is one having an end in Q whereas no domain of the δ chain can be a member of a chain having an end in Q. Thus $N_2, \cdots, N_n$ coincide with $M_2, \cdots, M_n$ at the points $s_2, \cdots, s_n$, but with $M_1, \cdots, M_{n-1}$ at $t_1, \cdots, t_{n-1}$. Let $M_2, \cdots, M_n$ follow $N_2, \cdots, N_n$ from $s_2, \cdots, s_n$ to $t_1, \cdots, t_{n-1}$. M_2 hence meets M_1 first at t_1 as t_1 is the first point of M_1 on the boundary of a c-domain of F_{12} which has no point of Q on its boundary. As s_n is the last end of any chain having an opposite end in Q, N_n and with it M_n as diverted leaves the original position of M_n at s_n. $N_2, \cdots, N_n$ having been independent so are $M_2, \cdots, M_n$ as now diverted.

In the next section of this proof it will be necessary to make various, even an infinite number of, switch and pocket constructions simultaneously. In what ways these constructions may affect each other is accordingly an essential problem. Various observations follow.

On Switches. (1) Any switch and its associated chains occupy on both (M_1) and (M_n) more than one point, for in $/s_nt_1/$ if $s_1 = t_1$ and $(as_1) \subset M_1$ then $\Gamma_1 \times (as_1) \neq 0$, and if $s_n = t_n$ and $(t_nb) \subset M_n$ then $\Delta_{n-1} \times (t_nb) \neq 0$.*

(2) Given $/s_nt_1/$, no point of (at_1) arc of M_1 is joined by any chain of F_{1n} to a point of (s_nb) arc of M_n. Also given $\backslash s_1t_n\backslash$ no point of (at_n) is so joined to one of (s_1b). These follow from the nature of the sets.

(3) If among the point pairs $s_1, s_1', t_1, t_1', s_n, s_n', t_n, t_n'$ one pair is non-coincident then $/s_nt_1/$ and $/s_n't_1'/$ have at most n points and these all invariant in common. This is the argument for one case. Suppose t_1' follows t_1 on (M_1); then s_1' does not precede t_1 as $(s_1't_1')$ is free. If s_n' were to precede t_n, then a point, t_1, preceding t_1' would be joined by a chain to a point,

* Notation here is as in the discussion immediately preceding III(c).

t_n, following s_n' which contradicts (2). But then the links of the chain γ' joining s_1' to s_n' follow or at most coincide with those of δ joining t_1 to t_n. Hence the switches have in common at most the points $s_i'=t_i$ $(i=1, 2, \cdots, n)$ which are invariant.

The same thing is true of $/s_nt_1/$ and $\diagdown s_1't_n'\diagup$ if t_n' follows t_n, or s_n' follows s_n, or if s_1' precedes s_1, or t_1' precedes t_1.* When t_n' follows t_n the proof is this. As $(s_n't_n')$ is free s_n' does not precede t_n, and if $s_n' = t_n$ then $t_n \neq s_n$ by observation (1). But then if s_1' precedes t_1 a point, namely s_1' preceding t_1, is joined to a point, namely s_n' following s_n. This contradicts (2) and the statement follows as above.

(4) From (3) it appears that in any set of switches of the same type the switch construction may be made simultaneously as any two have in common only invariant points, the two sets of diverted arcs having in no wise interfered while changing position. Also from (3) it follows that two switches of different type when determined by pairs of corresponding intervals may have the switch construction made independently in both, as if $C\ D$ determines $/s_nt_1/$ and $C'\ D'$ following $C\ D$ determines $\diagdown s_1't_n'\diagup$ then some chain from M_n must have ends on M_1 following C and preceding C' so $C \times C' = 0$, and as t_n does not follow D while t_n' follows D' t_n' follows t_n, even though D and D' reduce to coincident points.

(5) Only a or b can be the limit of an infinite set of switches, for among any infinite set must be an infinite subset of the same type and between any two of the same type is a chain; then apply I(c).

On Pockets. (1) In $/s_nt_1/$ if (s_1t_1) is free the switch contains a pocket, otherwise a given switch and pocket have only invariant common points as between the two lies a chain.

(2) Pockets of the first of the four sorts enumerated are simple and cause no difficulties. Given then a set of pockets made up of the other sorts. Such a set if infinite can have only a or b as set limit. If between any two of the set there is a chain then any two have at most invariant common points and in all of the sets simultaneously and without interference the pocket construction can be made, as it does not affect invariant points. What then is the relation between two pockets not having a chain between them? Evidently one must depend on a maximum free segment (l_1f_1) of (M_1), the other on

* Other relations between switches of these types are possible and for these the statement is not necessarily true as may be seen by drawing a figure, but for using switches of these sorts fortunately no need arises.

one $(l_n'f_n')$ of (M_n). But the γ chain of the first pocket precedes the second pocket and the γ' chain of the second precedes the first, so these two chains must coincide. Similarly the δ and δ' chains must coincide. Hence $l_j = l_j'$ $(j = 1, 2, \cdots, n)$ and $f_j = f_j'$. The common points of the b-boundary arc $l_1l_2, \cdots, l_{n-1}l_n$ of the γ chain and the a-boundary arc $f_1f_2, \cdots, f_{n-1}f_n$ of the δ chain are not invariant, as the two pockets have no chain between. If there is more than one such point the two pockets have no common point, if there is one such point they have it in common, if there is no such point actually there is but one pocket of sort (4) (see pocket classification), but it will be convenient still to look upon this set as two pockets. Now if $(l_1l_n) \times (f_1f_n)$, these symbols representing the chain boundary arcs in question, is vacuous or but a single point, evidently pocket constructions in each of the pockets may interfere, fig. 2(d). Such a pair of pockets as well as the free segments determining them will be said to be *opposite*. Evidently their sum is a switch of either type, $/l_nf_1/$ or $\diagdown l_1f_n \diagdown$.

(3) In section 4 pocket constructions will be used when available to divert M_1 or M_n from one or other but not both of the intervals in each pair of a given set of corresponding intervals. Among these intervals certain ones by being contained in maximum free segments, for example (l_1f_1), will identify an associated set of such segments and pockets. Suppose in fact that $(l_1f_1) \supset C_{i_s} + \cdots + C_u + \cdots + C_{i_t}$ $(i_s \leq u \leq i_t)$ but no others of the set, while (l_nf_n) the opposite of (l_1f_1), if it exists, contains $D_{j_s} + \cdots + D_v + \cdots + D_{j_t}$ $(j_s \leq v \leq j_t)$. The circumstances under which pocket constructions may and will be subsequently used to divert M_1 or M_n and the other arcs affected from one of each pair of corresponding intervals in the set is here summarised as condition (P).*

(P): *Among the given set of corresponding intervals at least one of each pair is free, and one at least of the free ones of each pair is in a maximum free segment, as* (l_1f_1), *which either* (1) *has no opposite among the pockets determined by the intervals of the set* (that is (l_nf_n) fails to exist or fails to contain intervals of the set), *or* (2) *has an opposite in which a pocket construction can be made independent of that in itself, or* (3) *has an opposite containing intervals corresponding only to those in itself or an opposite containing every interval to which an interval in itself corresponds,* (that is $i_s \leq j_s$ and $j_t \leq i_t$, or $j_s \leq i_s$ and $i_t \leq j_t$.).

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* This condition can be made some what less restrictive but not without difficulty out of proportion to the benefit accruing.

These conditions not being satisfied, if $j_s < i_s$ and therefore $j_t < i_t$, the sum of the pockets will be taken as the switch $/l_n f_1/$ or $/s_n t_1/$; whereas if $i_s < j_s$ and hence $i_t < j_t$ the sum will be taken as $\backslash l_1 f_n \backslash$ or $\backslash s_1 t_n \backslash$.

The next section will be devoted to the proof of the theorem mentioned at the end of section 2.

4. THEOREM. *If none of the pairs of corresponding intervals of M_{1n} is of type* 17 *then F contains $n+1$ independent arcs from a to b.*

Assuming that none of the pairs is of this type construction of the $n+1$ arcs, to be designated as $L_1, L_2, \cdots, L_n, L_{n+1}$, will be attempted. Convenience divides the argument into the following three cases; in case 1 it will be assumed that the arc of II(c) does not occur upon C_0 or D_0, and that the entire set of corresponding intervals satisfies (P); in case 2 that the same arc is not on C_0 or D_0 but that (P) is not satisfied; in case 3 it will be assumed that the arc of II(c) is on C_0.

Proof: Case 1. Let $G = I + K + M_{1n}$. The topological properties of G are precisely those of any set F_{jj+1}, in fact by a transformation of the type described in II(e) G is converted into such a set. Let $\sum_{i=-\infty}^{+\infty} C_i$ be the set of intervals of M_1, and $\sum_{i=-\infty}^{+\infty} D_i$ those of M_n. Subsets $[C]$ and $[D]$ will now be chosen as follows from these. For each j let $[C]$ contain C_j if (1) C_j is free but D_j is not, or if (2) both C_j and D_j are free and the pocket determined by C_j either has no opposite in the set of pockets determined by $\sum_{i=-\infty}^{+\infty} D_i$ or has one in which the pocket construction may be made independently of one in itself or has one containing only intervals which correspond to intervals contained in itself; and let $[D]$ contain D_j if D_j is free and C_j is not or if both C_j and D_j are free and either the pocket determined be D_j has no opposite among those determined by $\sum_{i=-\infty}^{+\infty} C_i$ or has an opposite in which a pocket construction can be made independently of one in itself or an opposite containing only intervals corresponding to those in itself but not all of them. As the entire set of intervals satisfies (P) this analysis for each pair of intervals puts C_j into $[C]$, or D_j into $[D]$, or both. Moreover, pocket constructions may be made simultaneously in all of the pockets determined by all the intervals of $[C]$ and $[D]$, and independently.

Let the cross domains of G be $\cdots \phi_{-p}, \cdots, \phi_{-2}, \phi_{-1}, \phi_0, \phi_1, \phi_2, \cdots, \phi_q, \cdots$, ($\phi_{-1} = \phi_0$ if $K_0 = 0$). From the definition of K and from II(e)

$\sum_{i=-\infty}^{+\infty} \Phi_i \times (M_1) = \sum_{i=-\infty}^{+\infty} C_i$ and $\sum_{i=-\infty}^{+\infty} \Phi_i \times (M_n) = \sum_{i=-\infty}^{+\infty} D_i$. Let $[\Phi]$ be the subset of $\sum_{i=-\infty}^{+\infty} \Phi_i$ for which $\sum_{i=-\infty}^{+\infty} \Phi_i \times (M_1) = [C]$, then by the classification determining $[C]$ and $[D]$ $(\sum_{i=-\infty}^{+\infty} \Phi_i - [\Phi]) \times (M_n) \subset [D]$. Following III(b) divert the arcs $M_1, \cdots, M_n$ from the original positions of M_1 and M_n through the pockets associated with the intervals of $[C]$ and $[D]$. The resulting point sets are still independent by reason of an assumption made throughout this case, and are arcs owing to that property for pockets corresponding to (5) for switches. Call them $L_1, \cdots, L_n$; evidently $F_{j-1j+1} \supset L_j (j = 2, 3, \cdots, n-1)$, $F_{12} \supset L_1$, $F_{n-1n} \supset L_n$,; and $L_1 \times [C] = 0$, $L_n \times [D] = 0$, $L_i \times L_j = a + b$ $(i \neq j)$.

In G L_{n+1} will now be constructed. To do this use I(d) in G, $j = 1$ $i = 1$, and $Q = \sum_{i=-\infty}^{+\infty} C_i - [C]$. The set Q_2, which is clearly on (M_n) not on (M_2), will then, since $\sum_{i=-\infty}^{+\infty} C_i - [C] = (\sum_{i=-\infty}^{+\infty} \Phi_i - [\Phi]) \times (M_1)$, be equal to the set $(\sum_{i=-\infty}^{+\infty} \Phi_i - [\Phi]) \times (M_n)$, that is $Q_2 \subset [D]$. As N_2 constructed by I(d) is for any Q such that $N_2 \times (M_1) \subset \sum_{i=-\infty}^{+\infty} \Phi_i \times (M_1)$ and $N_2 \times (M_n) \subset \sum_{i=-\infty}^{+\infty} \Phi_i \times (M_n)$; for this particular Q, $N_2 \times (M_1) = [C]$ and so $N_2 \times (L_1) = 0$; also $N_2 \times (M_n) \subset Q_2$, hence $N_2 \times (M_n) \subset [D]$ and so $N_2 \times (L_n) = 0$. Likewise $N_2 \times (\sum_{j=2}^{n-1} (L_j)) = 0$ as $N_2 \times I_{1n} = 0$ whereas $I_{1n} \supset \sum_{j=2}^{n-1} (L_j)$. The set N_2 is clearly an arc; let it be L_{n+1}. But then $L_j (j = 1, 2, \cdots, n+1)$ is a set of arcs in F from a to b contradicting the theorem in this case since independent and $n+1$ in number.

Case 2: The suppositions on this case imply at least one pair of corresponding intervals belonging to type 8, or to type 9, or to types 1 to 7 with one or both free and subsets of maximal free segments determining pockets having opposites among the set of pockets determined by $\sum_{i=-\infty}^{+\infty} C_i$ and $\sum_{i=-\infty}^{+\infty} D_i$, the pocket constructions in an opposite pair not being independent and the pair itself being such that, (a) $j_s < i_s$ and $j_t < i_t$, or (b) $i_s < j_s$ and $i_t < j_t$. Both 8 and the free case with (a) imply a switch of the sort $/s_n t_1/$ with s_n preceding D of the determining pair and t_1 following C; the construction for these two cases is identical and hereafter the two will not be kept separate. Similarly 9 and the free case with (*b*) yield to the same treatment; in fact these two cases need not be considered at the outset at all since upon reflecting

F with respect to any straight line parallel to the straight line ab and then renumbering the arcs M_j in the reverse order, these two are the same as the first two. If a contradiction is obtained when a start is made with the first two one may similarly be derived for the other two. The procedure being the same whatever the corresponding interval pair upon which 8 or (a) occurs, suppose it to be $C_0 D_0$.

Let $/s_n t_1/$ be any switch determined by 8 or the free case with (a) in connection with $C_0 D_0$. As t_1 follows C_0 it follows some last interval C_{l_1-1} of M_1, $l_1 - 1 \geq 0$; and as s_n precedes D_0 it precedes some first interval D_m of M_n, $m \leq 0$. Deflect L_1 from M_1 into K_{l_1} through $a_{1 l_1}$; in the switch $L_2, \cdots, L_n$, whose construction is thus begun, move from $M_2, \cdots, M_n$ to $M_1, \cdots M_{n-1}$, L_2 first reaching M_1 at t_1 while L_n leaves M_n at s_n. Finally from any convenient point of K_m introduce L_{n+1} upon M_n at b_{nm}. The construction of the $n+1$ independent arcs is now begun. Their continuation depends entirely upon the nature of the remaining pairs of intervals and for convenience various sub-cases will now be distinguished. As the situations met while constructing the $n+1$ arcs to a are no more general than those met toward b, only the construction to b will be completed as that to a follows the same lines.

The following possibilities call for separate treatment: (1) there is no pair of corresponding intervals subsequent to $C_{l_1-1}\, D_{l_1-1}$; (2) the next or some subsequent pair implies a switch $\diagdown s_1' t_n' \diagdown$ with s_1' preceding C_{l_1}, t_1' following D_{l_1}; (3) the next or some subsequent pair implies a switch $/s_n' t_1'/$ with s_n' preceding D_{l_1} and t_1' following C_{l_1}; (4) neither the next nor any subsequent pair implies a switch; (5) there exist subsequent pairs which imply switches but none of these has the property explicitely added in (2) or (3).

Case 2 (1). Observe in this connection that t_1 is different from b; this is obvious if the switch is determined by 8 while if determined by (a) $t_1 = b$ would mean $(s_1 b)$ on M_1 and thus $(s_n b)$ on M_n were both free and so $i_t = j_t$, a contradiction. But if $t_1 \neq b$, since no pair of corresponding intervals follows $C_{l_1-1} D_{l_1-1}$, $K_{l_1} = B \neq 0$. But L_1 being thus in B may be drawn through it to b, $L_2, \cdots, L_{n+1}$ meanwhile following $M_1, \cdots, M_n$ independently of each other and of L_1 to b, thus in this case completing the $n+1$ arcs to b.

Case 2 (2). Special circumstances attend the occurrence of this case also. Owing to note (2) on switches s_1' must not precede t_1 and so, since C_{l_1-1} precedes t_1 and C_{l_1} follows s_1', $a_{1 l_1} \neq b_{1 l_1}$. This being the case proceed as follows. Let $\diagdown s_1' t_n' \diagdown$ be the last such switch with s_1' preceding $C_{1 l_1}$ and

let the last interval of a corresponding pair preceding t_n' be D_{l_2-1}, $l_2 > l_1$. In the switch $\backslash s_1' t_n' \backslash$ let the arcs $L_2, \cdots, L_n$ drop back from $M_1, \cdots, M_{n-1}$ to $M_2, \cdots, M_n$; let L_1 return upon M_1 from K_{l_1} through b_{1l_1} the first point of C_{l_1} and thus independent of L_2 which leaves M_1 at a preceding point namely s_1'; finally let L_{n+1} follow M_n to a_{nl_2} and there pass into K_{l_2} quite independent of L_n which first reaches M_n at t_n' subsequent to a_{nl_2} which is the last point

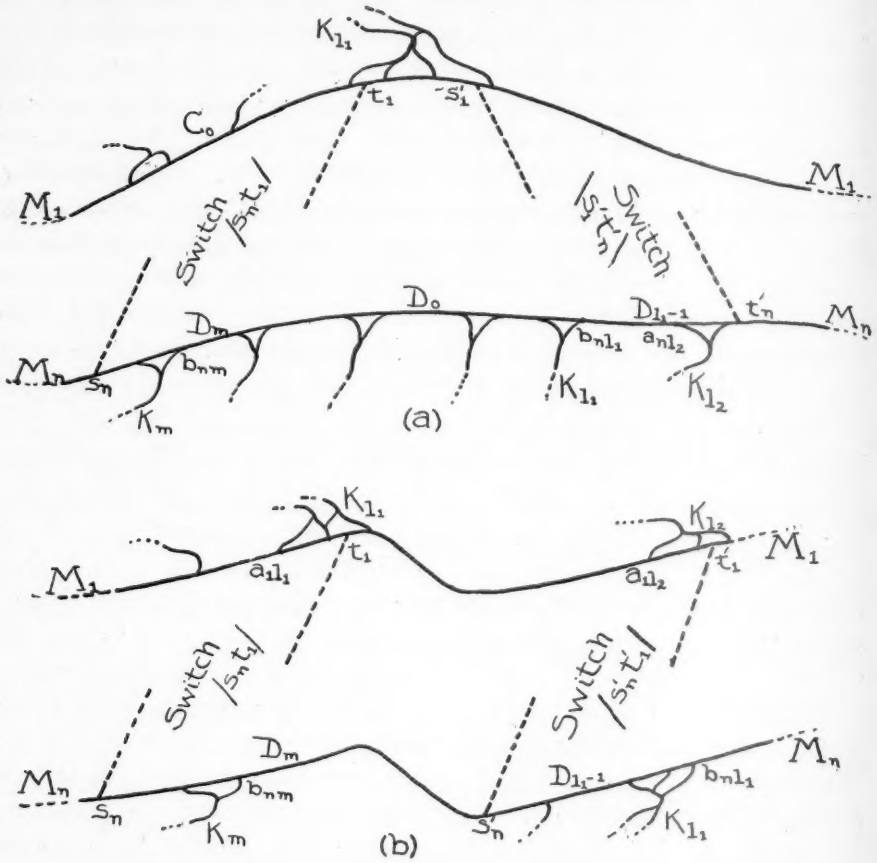

FIG. 4.

of D_{l_2-1}, fig. 4(a). The $n+1$ independent arcs have now been carried on into the next pair of corresponding intervals, or a pair subsequent to these.

Case 2 (3). This case is slightly more general than the two preceding as it imposes no conditions upon I or K. Two suppositions requiring somewhat different handling will be treated separately. As $/s_n' t_1'/$ will at the outset be selected the last switch of this sort having s_n' before D_{l_1}. Suppose

s_n' does not precede D_m. Let C_{l_2-1} be the last interval of M_1 preceding t_1', $l_2 > l_1$. The switch construction can be employed immediately. Let L_3, $\cdots$, L_{n+1} follow $M_2, \cdots, M_n$ from their last points in the switch $/s_n t_1/$ to their first in $/s_n' t_1'/$, in so doing clearly they remain stationary or approach b owing to note (3) on switches, in particular are these statements true of L_{n+1} as s_n' does not precede D_m. Let them in $/s_n' t_1'/$ then be raised from $M_2, \cdots, M_n$ to $M_1, \cdots, M_{n-1}$, L_{n+1} leaving M_n at s_n' previous to point b_{nl_1}, L_3 reaching M_1 first at t_1' after a_{1l_2}; take L_2 out into K_{l_2} through a_{1l_2}; and finally bring L_1 in from K_{l_1} upon M_n through b_{nl_1}. $L_1, \cdots, L_{n+1}$ have meanwhile remained independent and have been introduced upon the next or a subsequent pair of intervals, fig. 4 (b).

Before undertaking the next case the previous one must be completed for the assumption s_n' precedes D_m; this may certainly occur as $/s_n t_1/$ was chosen as any switch having s_n before D_0, t_1 after C_0; it can not occur if $/s_n' t_1'/$ is the last switch having s_n before D_0, and consequently can not occur for $/s_n'' t_1''/$ after $/s_n' t_1'/$ nor $\diagdown s_1'' t_n'' \diagup$ after $\diagdown s_1' t_n' \diagup$ in constructing after cases 2 (2) and the first part of 2 (3) as $/s_n' t_1'/$ and $\diagdown s_1' t_n' \diagup$ were selected as the last switches satisfying other requirements as imposed there. It is undesirable to pick $/s_n t_1/$ with the thought of avoiding this case as doing that makes the construction toward b less general than that toward a. However, the situation s_n' precedes D_m affects the fate of only one of the arcs involved and that arc is L_{n+1}. When the switch construction is made in $/s_n' t_1'/$ $L_1, \cdots, L_n$ behave as above, but L_{n+1} instead of remaining where it is or approaching b must turn from b_{nm} toward a along M_n until it reaches s_n' and there be acted upon by the switch. Two difficulties seem to arise. It looks as if L_{n+1} might have to go as far as s_n before encountering s_n' for possibly s_n coincides with t_n; the independence of the arcs would thus be lost as L_n already occupies s_n. This difficulty vanishes upon referring to note (1) on switches where it is seen that if $s_n = t_n$ then Δ_{n-1} has a point on M_n following t_n which s_n' can not precede. The other difficulty is this. Perhaps in the next step a switch $/s_n'' t_1''/$ will need to be used which has s_n'' also preceding D_m so that L_1 to reach it will have to approach a from b_{1l_1} thus encountering L_{n+1} at b_{nm}; this is impossible because $/s_n' t_1'/$ was the last switch for which s_n' preceded D_{l_1} so $/s_n'' t_1''/$ must have s_n'' on or following D_{l_1}. Or perhaps a pocket required on D_{l_2} depends on a maximum free segment having an end not following b_{nm} so that L_1 might have to come to this point to enter the pocket construction thus colliding with L_{n+1}; this is impossible owing to the situation on $C_{l_1} D_{l_1}$ for D_{l_1} either has a chain actually ending upon it (e. g. when C_{l_1}

D_{l_1} are of type 8) or is contained in a pocket which with its opposite forms a switch $/s_n't_1'/$ which is followed by a chain from t_n to t_1, both which circumstances prevent any pockets on D_{l_2} having an end preceding D_{l_1} much less one not following b_{nm}. The $n+1$ arcs are thus successfully extended in such a way that subsequent construction along similar lines will not be interfered with.

Case 2 (4). The statement of this case indicates that among the pairs of corresponding intervals $C_{l_1}D_{l_1}$, $C_{l_1+1}D_{l_1+1}, \cdots$ no pair is of types 8 or 9 and no pair contains only free intervals exhibiting (a) or (b) as described at the beginning of this section. Thus this set of intervals satisfies condition (P). Represent those of the set on M_1 by $\sum_{i=l_1}^{+\infty} C_i$ and those on M_n by $\sum_{i=l_1}^{+\infty} D_i$ and using the same criterion as in case 1 select from these two summations the subsets $[C]$ and $[D]$. Now in G with Q on M_1 equal to $\sum_{i=l_1}^{+\infty} C_i - [C]$ construct N_2 by I(d). Since for any Q on M_1 and N_2 in G, $N_2 \times (M_1) = \sum_{i=-\infty}^{+\infty} C_i$ and $N_2 \times (M_n) \subset \sum_{i=-\infty}^{+\infty} D_i$ and since $b_{1l_1}b \times \sum_{i=-\infty}^{+\infty} C_i = \sum_{i=l_1}^{+\infty} C_i$ and $b_{nl_1}b \times \sum_{i=-\infty}^{+\infty} D_i = \sum_{i=l_1}^{+\infty} D_i$ for this particular Q $N_2 \times b_{1l_1}b) = [C]$ and $N_2 \times b_{nl_1}b) = [D]$, $b_{1l_1}b)$ and $b_{nl_1}b)$ representing arcs of M_1 and M_n respectively which lack the end point b. But the intervals of $[C]$ and $[D]$ are subsets of maximal free segments of M_{1n} all following t_1 and t_n in the pockets determined by which the pocket construction can be made simultaneously and independently. Once the arcs $L_2, \cdots, L_{n+1}$ can be brought to position at $t_1, t_2, \cdots, t_{n-1}, t_n$ without points upon $[C]$ and $[D]$ they may be subjected to this construction and carried on independently to b in such a way that $L_2 \times [C] = 0$ and $L_{n+1} \times [D] = 0$. But the switch construction has already left $L_2, \cdots, L_n$ at $t_1, \cdots, t_{n-1}$ respectively, so the only question is as to the whereabouts of L_{n+1}. The arc L_{n+1} had been brought to M_n at b_{nm}, the first point of D_m and clearly preceding any interval of $[D]$, from K_m. If the first pocket required by $[D]$ is determined by the maximal free segment (pq) and $b_{nm} = p$ then construction may proceed without any trouble; if p follows b_{nm} then draw L_{n+1} along M_n until p is reached; if p precedes b_{nm} * draw L_{n+1} along M_n toward a from b_{nm} until it meets p. As q follows b_{nl_1}, it follows b_{nm}, so no difficulties arise. The pocket construction can then be used and $L_2, \cdots, L_{n+1}$ taken on independently to b, with $L_2 \times [C] = 0$ and

* Of course p does not precede t_n and if $s_n = t_n$ it can not coincide with t_n. Compare the situation here with that in the last paragraph of case 2 (3).

$L_{n+1} \times [D] = 0$. But N_2 in passing from the boundary of ϕ_{l_1}, whose limits on M_1 and M_n are C_{l_1} and D_{l_1} respectively, to the boundary of ϕ_{l_1-1} must pass through a point or limit point of K_{l_1} which is clearly not on L_2 or L_{n+1}. Now L_1 when last heard from was in K_{l_1}; join it then in K_{l_1} to any convenient point of N_2 and let it follow N_2 to b. The construction of $n+1$ independent arcs has thus been in this case as in case 1 completed to b.

Case 2 (5). Let $C_r D_r$ be the first pair of corresponding intervals $(r > l_1)$ which belong to types 8 or 9 or form with the pairs $C_{l_1} D_{l_1}$, $C_{l_1+1} D_{l_1+1}$, $\cdots$, $C_{r-1} D_{r-1}$ a set not satisfying condition (P). The pair implies a switch of type $/s_n' t_1'/$ or $\diagdown s_1' t_n' \diagdown$, suppose in order to make the discussion slightly less abstract, and for this reason only as the argument for either case follows similar lines, that the switch is $/s_n' t_i'/$. Let the symbol represent the last switch for which s_n' precedes D_r, and suppose D_{l_2} is the first that s_n' precedes $l_1 < l_2 \leq r$. The pairs of corresponding intervals $C_{l_1} D_{l_1}, \cdots, C_{l_2-1} D_{l_2-1}$ satisfy the condition (P). Form the subset $[C]$ of $\sum_{i=l_1}^{l_2-1} C_i$ and the subset $[D]$ of $\sum_{i=l_1}^{l_2-1} D_i$. Let $Q = \sum_{i=l_1}^{l_2-1} C_i - [C]$ and proceed precisely as in (4) to construct N_2. When N_2 is constructed join L_1 to it from K_{l_1} and let L_1 follow N_2 toward b until it reaches the first limit point of K_{l_2}, that is necessarily $a_{1 l_2}$ or $a_{n l_2}$, and then carry L_1 into K_{l_2}, out of the way. Meanwhile as in 2 (4) bring $L_2, \cdots, L_{n+1}$, with precisely the same circumspection as regards L_{n+1}, on until they arrive at the switch $/s_n' t_1'/$ having on the way avoided all the intervals of $[C]$ and $[D]$ and thus independent of L_1. In the switch then execute exactly the same maneuver as in case 2 (3). The $n+1$ independent arcs are thus introduced upon an interval pair subsequent to the pair $C_{l_1-1} D_{l_1-1}$.

In summary upon case 2 notice that in subcases (1) and (4) the problem is complete, while in the other cases the work paused at the point where the next remark would be, "there are now five possibilities," seen at a glance to be identical with the five just handled for the interval pair $C_{l_1} D_{l_1}$ and disposable in essentially the same way for the pair, say, $C_{l_2} D_{l_2}$, even although here they may follow a switch construction $\diagdown s_1' t_n' \diagdown$ rather than merely $/s_n t_1/$. So continue the process. Either eventually a pair of intervals is reached exhibiting case 2 (1) or (4) at which the construction ends, or it runs on through an infinite number of stages bringing $n+1$ independent arcs indefinitely near to b. To see that in the limit these arrive as independent arcs at b consult II(d), note (5) on switches, and the first few sentences of (2) on pockets.

Accordingly if the limit points of J are not on C_0 (or D_0) one pair of corresponding intervals is of type 17, as otherwise from a to b in F there are $n+1$ independent arcs.

Case 3. Suppose then K_0 is vacuous * and the limits of J occur on C_0. Omit J from F. Then $F-J$ contains on M_{1n} a pair of intervals of type 17. Restore J to the set. It will now be shown that if in $F-J$ the intervals $C_0\ D_0$ are the only corresponding ones of type 17 and that all the limits of the joining chains on C_0 occur between the extreme limits of J so that D_0 is no longer joined to either of the intervals into which as mentioned in II(e) J divides C_0, then a contradiction arises—there are $n+1$ arcs from a to b in F. For assume these conditions. For J the notation attached to K_0 will be used as far as it applies. There is at least one chain from D_0 to $(a_{10}\, b_{10})$ and from any convenient point of that constituent of J which connects a_{10} and b_{10} and any convenient points invariant and n in number one from each of the arcs $M_1, \cdots, M_n$ and common to the successive links of such a chain $n+1$ arcs will now be constructed. When the extra arc arrives in $K_m, m>0$, troubles cease, as no subsequent corresponding intervals are of type 17 and so previous methods apply.

Now D_0 is not free. Suppose C_0 is free. If the pocket containing it has no opposite among the pockets determined by the intervals $\sum_{i=0}^{+\infty} D_i$ or has an opposite for which $i_t > j_t$, pocket construction will be used depending upon the maximum free segment (pq) p preceding b_{10}, q following C_0, in fact following some last interval C_{m-1}. From the points indicated in the last paragraph extend $L_1, \cdots, L_n$ and let them be diverted from $M_1, \cdots, M_n$ only when in the pocket (pq) in which case let them follow its pocket construction; they are thus independent and $L_1 \times (pq) = 0$. From the chosen point of J bring L_{n+1} in through b_{10} upon M_1 which let it follow to a_{1m} last point of C_{m-1} from which it passes into $K_m (m>0)$. Farther construction now follows previous lines.

If C_0 is not in a pocket, or if when in one is in one which has an opposite with $i_t < j_t$ then a switch $\diagdown s_1\ t_n \diagdown$ is determined with s_1 on $(a_{10}\, b_{10})$ and t_n following $D_{m-1} (m>0)$. This switch drops the arcs $L_1, \cdots, L_{n-1}$ from the points at which they are begun to $M_2, \cdots, M_n$ L_1 leaving M_1 at s_1, L_{n-1} reaching M_n at t_n; let meanwhile L_n leave M_n at a_{nm} for $K_m (m>0)$ and let

* If K_0 is not vacuous the limits of J must occupy a subinterval of $a_{10}b_{10}$ or of $a_{n0}b_{n0}$ owing to the nature of the set K_0, see section 2.

L_{n+1} come in from J upon M_1 through b_{10}. As the $n+1$st arc is now in K_m instead of J nothing farther need be said.

The procedure toward a is similar, it being noted that although J redivides C_0 into C_{-1} and C_0 it does not reseparate D_{-1} and D_0 which still coincide. Thus there are $n+1$ arcs in F from a to b in this case also unless a pair of corresponding intervals is joined.

5. THEOREM I. *F posseses n complementary domains, of which one may be its unbounded complementary domain, $\lambda_{12}, \lambda_{23}, \cdots, \lambda_{n-1n}, \lambda_{n1}$, where λ_{jj+1} is a subset of I_{jj+1} and λ_{n1} is exterior to M_{1n}, such that:*

(1) *Λ_{jj+1} has at least one point in common with $\Lambda_{j-1\,j}$ and at least one in common with Λ_{j+1j+2} but none whatever in common with any of the others unless possibly a or b (subscripts reduced mod n), $(j=1, 2 \cdots n-1)$.*

(2) *all the points common to $\Lambda_{j-1\,j}$ and Λ_{jj+1} are on the arc M_j,*

(3) *at least one of the common points of $\Lambda_{j-1\,j}$ and Λ_{jj+1} is on (M_j), that is, is different from a and from b; this point will be designated as p_j.*

Proof: Any chain joining two corresponding intervals plus the domain of whose boundary these intervals is a subset form a set of domains satisfying these conditions.

Let any one of the points of $\Lambda_{j-1\,j} \times \Lambda_{jj+1} \times (M_j)$ be called p_j.

THEOREM II. *$F - \sum_{i=1}^{n} p_i$ consists of two mutually separated sets F_a and F_b where $F_a \supset a$ and $F_b \supset b$.*

Proof: Let c_{jj+1} be a point of λ_{jj+1}. As any boundary point of a complementary domain of a continuous curve is accessible from any point of the domain there is an arc $(p_j\, c_{jj+1}\, p_{j+1})$ in λ_{jj+1}. Draw all n of these arcs. As they are finite in number, join only at their ends, and join in an order determined by part (1) of theorem I above, the sum $\sum_{j=1}^{n} p_j\, c_{jj+1}\, p_{j+1}$ is a simple closed curve C, having only the points $\sum_{i=1}^{n} p_i$ in common with F. Moreover a and b are separated by this curve for suppose both are interior. Then b must be interior to one of the following simple closed curves with mutually

exclusive interiors $a\, m_1\, p_1\, c_{12}\, p_2\, m_2\, a$, $a\, m_2\, p_2\, c_{23}\, p_3\, m_3\, a, \cdots$, $a\, m_{n-1}\, p_{n-1}\, c_{n-1n}\, p_n\, m_n\, a$, $a\, m_n\, p_n\, c_{n1}\, p_1\, m_1\, a$ where m_j is some point of M_j between a and p_j. Suppose for instance that it lies in $a\, m_1\, p_1\, c_{12}\, p_2\, m_2\, a$. Then M_j $(j = 3, 4, \cdots, n)$ must to reach b from a cross $a\, m_1\, p_1$, or $p_1\, c_{12}\, p_2$, or $p_2\, m_2\, a$, none of which it may do; or else lie already within M_{12} entirely, which the preliminary ordering forbids. Similarly it is proved that not both a and b lie outside the simple closed curve. Thus F consists of two parts F_a and F_b, $F_a \supset a$ and $F_b \supset b$, one exterior to C and one interior, and of course the points $\sum_{i=1}^{n} p_i$. The theorem is thus seen to be proved.

THEOREM III. *$E - \sum_{i=1}^{n} p_i$ is the sum of two mutually separated sets E_a and E_b where $E_a \supset a$ and $E_b \supset b$.*

Proof: Suppose not. Then a and b are points of some maximal connected subset of $E - \sum_{i=1}^{n} p_i$ in which there can be drawn an arc W from one to the other. This arc cuts C in some point p, as a is separated from b by C. But p is not a point of F as it is not a point of $\sum_{i=1}^{n} p_i$. Proceed along W from p toward a and toward b until reaching the first points p_a and p_b in these directions which W has in common with F, these must be distinct as W is an arc, and must exist as a and b are in F and F is closed. But then $(p_a\, pp_b)$ is a part of some maximal connected subset of $E - F$ which has at least two limit points, for example p_a and p_b, in F. This is a contradiction.*

COROLLARY I. *Under the hypotheses of the general theorem E possesses a set of complementary domains having the property of Theorem I above.*†

THEOREM IV. *If among the subsets of the Continuous Curve E whose omission from the continuous curve separates it into two mutually separated sets E_a and E_b, $E_a \supset a$ and $E_b \supset b$ where a and b are given points, is one consisting of n distinct points but none consisting of fewer than n, then there is a set of n independent arcs in E from a to b and no set of independent arcs of E from a to b consists of more than n.*

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* W. L. Ayres, "Arc Curves and Basic Sets of a Continuous Curve."

† For F has such a set and then see W. L. Ayres, "Concerning Arc-curves and Basic sets of a Continuous Curve."

Proof: Obviously if the set of independent arcs from a to b in E were $n+1$ in number, $F - \sum_{i=1}^{n} p_i$ would not be separated between a and b, as at least one of the $n+1$ arcs would fail to contain any point of $\sum_{i=1}^{n} p_i$, so in this case $\sum_{i=1}^{n} p_i$ could not separate E in this way either. On the other hand, if the arc curve were to contain at most $n-j$ $(0 < j \leq n)$ independent arcs from a to b, then by the general theorem there would be $(n-j) < n$ points whose omission would separate E between a and b into two mutually separated sets one containing a and the other b. As this contradicts hypothesis, the arc curve of $a+b$ with respect to E contains for some construction n independent arcs from a to b but for no construction more than n.

THEOREM V. *Under the hypotheses of theorem IV there is a set of n complementary domains of E related to any chosen set of n separating points in the manner of the domains of theorem I and the points $\sum_{i=1}^{n} p_i$.*

Proof: For by theorem IV the arc curve of $a+b$ in E contains a set of n independent arcs but no set of more than n. Let a set of n be $M_1, \cdots, M_n$ ordered as heretofore and for each i let p_i be on (M_i), $\sum_{i=1}^{n} p_i$ being the chosen separating set; evidently one point of this set must be on each of the arcs M_i and no more than one can be. Now the set F_{jj+1} as previously described must be cut between a and b by the omission from it of $\sum_{i=1}^{n} p_i$ as it is a subset of F containing a and b and hence must also be cut by the omission of the points p_j and p_{j+1} which are the only ones of $\sum_{i=1}^{n} p_i$ which it contains. But then as before p_j and p_{j+1} are on the boundary of some bounded complementary domain of F_{jj+1}, hence also one of F like λ_{jj+1} and finally one of E. The set of domains thus determined is easily seen to possess the properties of those in theorem I.

THEOREM VI. *Either under the hypotheses of the general theorem or under the hypotheses of theorem IV, there is a simple closed curve C whose interior contains either a or b and such that $C \times E = \sum_{i=1}^{n} p_i$ where $\sum_{i=1}^{n} p_i$ is any set of n points whose omission divides E into two mutually separated sets one containing a and the other b.*

Proof: As the sets of n points dividing the curve E are identical with those dividing the curve F, the theorem follows upon the application of theorem V and the process of theorem II. It is to be observed that C can not enclose a or b chosen arbitrarily as the set of domains of theorem V may possibly not include the unbounded complementary domain of E.

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A Direct Treatment of Systems of Linear Differential Equations whose Coefficients have Uniform Singularities.

By J. A. Nyswander.

A complete theory of solution of linear differential equations of the form

$$\frac{d^n x}{dt^n} + \frac{p_1(t)}{t-t_0}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + \frac{p_n(t)}{(t-t_0)^n}x = 0,$$

in which the $p_i(t)$ are analytic about $t = t_0$, was given by Fuchs * about 1868. He treated exhaustively the various cases that can arise, classified with respect to the nature of the roots of the characteristic equation

$$f(\lambda) = \lambda(\lambda-1)\cdots(\lambda-n+1) + \lambda(\lambda-1)\cdots(\lambda-n+2)p_1(0) + \cdots + \lambda p_{n-1}(0) + p_n(0) = 0.$$

The more comprehensive problem of obtaining, under completely general hypotheses with respect to the roots of $f(\lambda) = 0$, a fundamental set of solutions for the system of differential equations

$$t(dx_i/dt) = \theta_{i1}(t)x_1 + \cdots + \theta_{in}(t)x_n \qquad (i = 1, \cdots, n),$$

was first solved by Horn about 1891. In his theory Horn employed a principle originally due to Frobenius † and later developed by Grünfeld.‡ The treatment given by Horn is based on the use of sequential transformations; the various cases that arise being classified in terms of the elementary divisors of the characteristic determinant $D(\lambda)$.

The present paper gives a direct and relatively simple treatment of the general problem, and shows how to obtain the solutions in all cases without recourse to either transformations or the theory of elementary divisors.

1. Consider the system of differential equations

$$(1) \qquad t(dx_i/dt) = \theta_{i1}(t)x_1 + \cdots + \theta_{in}(t)x_n \qquad (i = 1, \cdots, n)$$

* Horn, "Gewöhnliche Differentialgleichungen beliebiger Ordnung"; Sauvage, "Théorie Générale des Systèmes d'Équations Differentielles."

† *Crelle's Journal*, Vol. 74.

‡ Grünfeld, "Über die Integration eines Systems," *Denkschriften der Wiener Akademie*, Mathematisch-Naturwissenschaftliche Klasse, 1888.

in which the $\theta_{ij}(t)$ are analytic for $t = 0$ and hence may be written in the form

$$\theta_{ij}(t) = \sum_{k=0}^{\infty} \theta_{ij}^{(k)} t^k \qquad (i, j = 1, \cdots, n).$$

Suppose, also, that at least one $\theta_{ij}^{(0)} \neq 0$. It will be shown, first, that under certain conditions system (1) will have a solution of the form

$$x_i = t^\lambda \sum_{k=0}^{\infty} a_i^{(k)} t^k \equiv t^\lambda P_i(t : \lambda) \qquad (i = 1, \cdots, n) \tag{2}$$

in which the λ and $a_i^{(k)}$ are constants to be determined.

On substituting (2) in (1) there results the following relations, which must be identities in t if (2) is to be a solution of (1):

$$\lambda t^\lambda P_i(t:\lambda) + t^{\lambda+1} \frac{dP_i(t:\lambda)}{dt} = \sum_{j=1}^{n} \sum_{k=0}^{\infty} \theta_{ij}^{(k)} t^{\lambda+k} P_j(t:\lambda) \qquad (i = 1, \cdots, n). \tag{3}$$

On equating the coefficients of corresponding powers of t in (3), one obtains as sufficient conditions that (2) shall be a solution of (1).

$$\begin{cases} (\theta_{11}^{(0)} - \lambda) a_1^{(0)} + \theta_{12}^{(0)} a_2^{(0)} + \cdots + \theta_{1n}^{(0)} a_n^{(0)} = 0 \\ \theta_{21}^{(0)} a_1^{(0)} + (\theta_{22}^{(0)} - \lambda) a_2^{(0)} + \cdots + \theta_{2n}^{(0)} a_n^{(0)} = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \theta_{n1}^{(0)} a_1^{(0)} + \cdots + (\theta_{nn}^{(0)} - \lambda) a_n^{(0)} = 0, \end{cases} \tag{4}$$

$$\begin{cases} (\theta_{11}^{(0)} - \lambda - p) a_1^{(p)} + \cdots + \theta_{1n}^{(0)} a_n^{(p)} = F_1^{(p)}(a_j^{(0)}, \cdots, a_j^{(p-1)}) \\ \theta_{21}^{(0)} a_1^{(p)} + \cdots + \theta_{2n}^{(0)} a_n^{(p)} = F_2^{(p)}(a_j^{(0)}, \cdots, a_j^{(p-1)}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \theta_{n1}^{(0)} a_1^{(p)} + \cdots + (\theta_{nn}^{(0)} - \lambda - p) a_n^{(p)} = F_n^{(p)}(a_j^{(0)}, \cdots, a_j^{(p-1)}) \end{cases} \qquad (p = 1, 2, \cdots), \tag{5}$$

where

$$F_i^{(p)} = -\sum_{j=1}^{n} \sum_{k=1}^{p} \theta_{ij}^{(k)} a_j^{(p-k)} \qquad (i = 1, \cdots, n). \tag{6}$$

Equations (4) which are linear and homogeneous in the $a_j^{(0)}$ will have a non-trivial solution if and only if λ is a root of the characteristic equation

$$D(\lambda) = \begin{vmatrix} \theta_{11}^{(0)} - \lambda, & \theta_{12}^{(0)}, & \cdots, & \theta_{1n}^{(0)} \\ \theta_{21}^{(0)}, & \theta_{22}^{(0)} - \lambda, & \cdots, & \theta_{2n}^{(0)} \\ \cdots & \cdots & \cdots & \cdots \\ \theta_{n1}^{(0)}, & \theta_{n2}^{(0)}, & \cdots, & \theta_{nn}^{(0)} - \lambda \end{vmatrix} = 0 \tag{7}$$

We shall first dispose of the simplest case, in which the n roots of $D(\lambda) = 0$ are distinct and no two of the λ_j differ by an integer. For each λ_j at least one first minor in $D(\lambda_j)$ will be different from zero, and hence the solutions of equations (4) may be written in the form

$$a_i^{(0)} = c_j \eta_{ij}^{(0)} \qquad (i = 1, \cdots, n;\ j = 1, \cdots, n),$$

where the $\eta_{ij}^{(0)}$ are the first minor-cofactors in $D(\lambda_j)$ which arise in the solution of equations (4) by determinants.

In view of the hypothesis regarding the roots of $D(\lambda) = 0$, it is evident that

$$D(\lambda_j + p) \neq 0 \qquad (j = 1, \cdots, n;\ p = 1, 2, \cdots).$$

Thus equations (5) determine the $a_i^{(p)}$ $(i = 1, \cdots, n;\ p = 1, 2, \cdots)$ for each λ_j and consequently one obtains n solutions of system (1) of the form (2). These solutions can be written in the form

$$x_i = \sum_{j=1}^{n} c_j t^{\lambda_j} P_{ij}(t) \qquad (i = 1, \cdots, n;\ j = 1, \cdots, n),$$

where

$$P_{ij}(t) = a_{ij}^{(0)} + a_{ij}^{(1)} t + a_{ij}^{(2)} t^2 + \cdots.$$

The convergence of the solutions obtained in the present and later sections will be established in Section 7. It will be shown in Section 6, under entirely general hypotheses in regard to the nature of the roots of the characteristic equation, that the n solutions obtained by the method of the present paper always form a fundamental set.

2. *Solutions when two Roots of $D(\lambda) = 0$ differ by an Integer.* It may happen that two or more of the λ_j differ by an integer.

Suppose that $\lambda_1 - \lambda_2 = d$, where d is a positive integer, and that there are no other congruences (mod 1) among the λ_j. Assume also that λ_1 and λ_2 are simple roots. Starting with a possible solution of form (2), one arrives, as in section 1, at equations (4) and (5). Clearly $D(\lambda_1 + p) \neq 0$ $(p = 1, 2, \cdots)$ and consequently a solution of form (2) for $\lambda = \lambda_1$, is obtained exactly as in section 1.

In attempting to arrive at a solution of form (2) for $\lambda = \lambda_2$ one observes that $D(\lambda_2 + d) = D(\lambda_1) = 0$, and hence equations (5) are inconsistent for $\lambda = \lambda_2;\ p = d$ unless the corresponding $F_i^{(d)}$ satisfy certain linear relations. Designate by $D_i(\lambda + p)$ the result of replacing the elements of the ith column of $D(\lambda + p)$ by $F_1^{(p)}, \cdots, F_n^{(p)}$. Obviously equations (5) for $\lambda = \lambda_2$ and $p = d$ will be consistent if and only if

$$D_i(\lambda_2 + d) = 0 \qquad (i = 1, \cdots, n). \tag{8}$$

CASE I. Assume that relations (8) are satisfied. The values of the $a_i^{(d)}$ as obtained from (5) may be thought of as a sum of two terms, one carrying an arbitrary constant as a factor, the other an expression involving the $F_i^{(d)}$. It is readily seen that the terms involving the arbitrary constant are exactly the values previously found for the $a_i^{(0)}$ of (4) with $\lambda = \lambda_1$. Since these terms if retained in determining the $a_i^{(p)}$, $p = d+1, d+2, \cdots$ would give rise to the solution of system (1) already obtained for $\lambda = \lambda_1$, it is desirable to omit them in computing the $a_i^{(p)}$. For $p > d$, it is seen that $D(\lambda_2 + p) \neq 0$, and consequently a solution of (1) of the form (2) exists in Case I.

CASE II. Suppose that the $F_i^{(d)}$ do not satisfy relations (8). We shall show that the desired solution of system (1) is of the type

$$x_i = t^{\lambda}(Q_i(t) + P_i(t)\log t) \qquad (i = 1, \cdots, n), \tag{9}$$

where P_i and Q_i are of the form

$$P_i(t) = \sum_{k=0}^{\infty} a_i^{(k)} t^k \qquad (i = 1, \cdots, n),$$

$$Q_i(t) = \sum_{k=0}^{\infty} b_i^{(k)} t^k \qquad (i = 1, \cdots, n).$$

On substituting (9) in (1), it is seen that the resulting relations will be identities in t, $t = 0$ being excluded, provided that the $a_i^{(k)}$ and $b_i^{(k)}$ satisfy the following relations:

$$(\theta_{ii}^{(0)} - \lambda)a_i^{(0)} + \sum_{k=1}^{n} \theta_{ik}^{(0)} a_k^{(0)} = 0 \qquad (i = 1, \cdots, n;\ k \neq i), \tag{10}$$

$$(\theta_{ii}^{(0)} - \lambda - p)a_i^{(p)} + \sum_{k=1}^{n} \theta_{ik}^{(0)} a_k^{(p)} = F_i^{(p)}(a_j^{(0)}, \cdots, a_j^{(p-1)}) \qquad (i = 1, \cdots, n;\ k \neq i;\ p = 1, 2, \cdots), \tag{11}$$

$$(\theta_{ii}^{(0)} - \lambda - p)b_i^{(p)} + \sum_{k=1}^{n} \theta_{ik}^{(0)} b_k^{(p)} = a_i^{(p)} + F_i^{(p)}(b_j^{(0)}, \cdots, b_j^{(p-1)}) \qquad (i = 1, \cdots, n;\ k \neq i;\ p = 0, 1, 2, \cdots), \tag{12}$$

where

$$F_i^{(0)} \equiv 0 \qquad (i = 1, \cdots, n).$$

In order to determine the $a_i^{(p)}$ and $b_i^{(p)}$ satisfying (10), (11), and (12), start with the n cofactors $\eta_1(\lambda), \cdots, \eta_n(\lambda)$ of the elements $\theta_{n1}^{(0)}, \cdots, \theta_{nn}^{(0)} - \lambda$ in the determinant $D(\lambda)$. Assume, for simplicity, that the first $n-1$ rows of $D(\lambda_2)$ contain a non-vanishing first minor; then $a_i^{(0)} = \eta_i^{(0)}(\lambda_2)$ $(i = 1, \cdots, n)$ is a solution of (10). Equations (10) are homogeneous in the $a_j^{(0)}$, and hence will be satisfied for

$$(13) \qquad a_i^{(0)}(\lambda) = (\lambda - \lambda_2)\,\eta_i^{(0)}(\lambda) \qquad (i = 1, \cdots, n;\ \lambda = \lambda_2),$$

where the $a_i^{(0)}(\lambda)$ are to be regarded as functions of λ. Substitute (13) into the right members of (11) and solve for the $a_i^{(p)}$ $(p = 1, 2, \cdots)$, where again the $a_i^{(p)}(\lambda)$ are to be thought of as functions of λ. In view of (13) one can write

$$a_i^{(p)}(\lambda) = (\lambda - \lambda_2)\,\eta_i^{(p)}(\lambda) \qquad (i = 1, \cdots, n;\ p = 1, 2, \cdots, d-1).$$

Since $\lambda - \lambda_2$ is a factor of $D(\lambda + d)$ and likewise of each $F_i^{(d)}(a_j^{(0)}, \cdots, a_j^{(d-1)})$ $(i = 1, \cdots, n)$, it follows that equations (11) give n functions of λ, $a_i^{(d)}(\lambda)$ $(i = 1, \cdots, n)$, which, for $\lambda = \lambda_2$, yield a non-trivial solution of the equations

$$(\theta_{ii}^{(0)} - \lambda_2 - d)a_i^{(d)} + \sum_{k=1}^{n} \theta_{ik}^{(0)} a_k^{(d)} = F_i^{(d)} = 0 \qquad (i = 1, \cdots, n;\ k \neq i).$$

For all values of $p > d$ equations (11) give functions of λ, $a_i^{(p)}(\lambda)$ $(i = 1, \cdots, n)$ not all of which are zero for $\lambda = \lambda_2$. Thus $a_i^{(p)}(\lambda_2)$ $(i = 1, \cdots, n;\ p = d, d+1, \cdots)$ satisfy equations (4) and (5) with $\lambda = \lambda_1$, it being noted that the $a_i^{(d)}(\lambda_2)$ satisfy (4); the $a_i^{(d+1)}(\lambda_2)$ satisfy (5) for $p = 1$, and so on. It is thus apparent that

$$x_i = [t^{\lambda} P_i(t : \lambda)]_{\lambda=\lambda_2} \qquad (i = 1, \cdots, n),$$

is a solution of system (1) and differs by a constant factor only from the solution already obtained, for $\lambda = \lambda_1$.

If we choose

$$(14) \qquad b_i^{(p)} = [\partial a_i^{(p)}(\lambda)/\partial\lambda]_{\lambda=\lambda_2} \qquad (i = 1, \cdots, n;\ p = 1, 2, \cdots),$$

it is easily shown that equations (12) are satisfied. For $p < d$ one has

$$b_i^{(p)} = [(\partial/\partial\lambda)(\lambda - \lambda_2) - \eta_i^{(p)}(\lambda)]_{\lambda=\lambda_2} = \eta_i^{(p)}(\lambda_2) \qquad (i = 1, \cdots, n).$$

Also for $p \leqq d$ equations (11) enable one to write the identities,

$$(15) \quad (\theta_{ii}^{(0)} - \lambda - p)a_i^{(p)}(\lambda) + \sum_{k=1}^{n} \theta_{ik}^{(0)} a_k^{(p)}(\lambda) \overset{\lambda}{\equiv} (\lambda - \lambda_2) F_i^{(p)}(\eta_j^{(0)}(\lambda), \cdots, \eta_j^{(p-1)}(\lambda)) \qquad (i = 1, \cdots, n;\ k \neq i).$$

After differentiating (15) with respect to λ and setting $\lambda = \lambda_2$ there results

$$(16) \quad (\theta_{ii}^{(0)} - \lambda_2 - p)(\partial a_i^{(p)}(\lambda)/\partial\lambda)_{\lambda=\lambda_2} + \sum_{k=1}^{n} \theta_{ik}^{(0)} (\partial a_k^{(p)}(\lambda)/\partial\lambda)_{\lambda=\lambda_2} = a_i^{(p)} + F_i^{(p)}[\eta_j^{(0)}(\lambda_2), \cdots, \eta_j^{(p-1)}(\lambda_2)] \qquad (i = 1, \cdots, n;\ k \neq i),$$

from which it is observed that the values of the $b_i^{(p)}$ in (14) satisfy (12) for $p \leqq d$. Note that the $a_i^{(p)}(\lambda_2) = 0$ for $p = 1, 2, \cdots, d-1$. If $p > d$ equations (11) give the identities

$$(17)\quad \begin{cases} (\theta_{ii}^{(0)} - \lambda - p)a_i^{(p)}(\lambda) + \sum_{k=1}^{n} \theta_{ik}^{(0)} a_k^{(p)}(\lambda) \overset{\lambda}{\equiv} F_i^{(p)}(a_j^{(0)}(\lambda), \cdots, a_j^{(p-1)}(\lambda)) \\ \quad \overset{\lambda}{\equiv} -\sum_{j=1}^{n} \sum_{k=1}^{p-d} \theta_{ij}^{(k)} a_j^{(p-k)}(\lambda) - (\lambda - \lambda_2) \sum_{j=1}^{n} \sum_{k=p-d+1}^{p} \theta_{ij}^{(k)} \eta_j^{(p-k)}(\lambda) \\ \qquad\qquad (i = 1, \cdots, n;\ k \neq i). \end{cases}$$

On differentiating (17) with respect to λ and setting $\lambda = \lambda_2$, one obtains

$$\begin{aligned} &(\theta_{ii}^{(0)} - \lambda_2 - p)(\partial a_i^{(p)}(\lambda)/\partial\lambda)_{\lambda=\lambda_2} + \sum_{k=1}^{n} \theta_{ik}^{(0)} (\partial a_k^{(p)}(\lambda)/\partial\lambda)_{\lambda=\lambda_2} \\ &= a_i^{(p)}(\lambda_2) - \sum_{j=1}^{n} \sum_{k=1}^{p-d} \theta_{ij}^{(k)} b_j^{(p-k)}(\lambda_2) - \sum_{j=1}^{n} \sum_{k=p-d+1}^{p} \theta_{ij}^{(k)} b_j^{(p-k)}(\lambda_2) \\ &= a_i^{(p)}(\lambda_2) + F_i^{(p)}(b_j^{(0)}(\lambda_2), \cdots, b_j^{(p-1)}(\lambda_2)) \\ &\qquad\qquad (i = 1, \cdots, n;\ k \neq i). \end{aligned}$$

Thus it has been shown that (14) yield the values of the $b_i^{(p)}$ satisfying (12) for $p = 0, 1, 2, \cdots$, and consequently (9) gives as the solution of system (1) associated with $\lambda = \lambda_2$,

$$x_i = t^{\lambda_2}[(\partial P_i(t:\lambda)/\partial\lambda) + P_i(t:\lambda)\log t]_{\lambda=\lambda_2} \qquad (i = 1, \cdots, n).$$

The methods of this section (Case II) have been presented in considerable detail because of their basic importance in the more general cases treated in later sections.

3. *Solutions corresponding to a Group of Roots of $D(\lambda) = 0$ mutually Congruent* (mod 1). Let $\lambda_1, \cdots, \lambda_e$ be e simple roots of $D(\lambda) = 0$ such that

$$(18)\qquad \lambda_i - \lambda_j = d_{ij} \qquad (i, j = 1, \cdots, e;\ i < j),$$

where the d_{ij} are positive integers. Suppose further that no other root of $D(\lambda) = 0$ differs from any one of the foregoing roots by an integer. A solution of system (1) of the form

$$(19)\qquad x_i = t^{\lambda_1} P_{i1}(t) \qquad (i = 1, \cdots, n),$$

is obviously obtainable by the method of Section 1. Solution (19) may, in

exact analogy with the scheme of Section 2, Case II, be represented by each of the e forms

$$x_i = t^{\lambda\sigma}[P_{i\sigma}(t:\lambda)]_{\lambda=\lambda_\sigma} \qquad (i=1,\cdots,n;\ \sigma=1,\cdots,e), \tag{20}$$

where the $P_{i\sigma}(t:\lambda)$ are power series in t which will now be defined. Let the constant term of $P_{i\sigma}(t:\lambda)$ be $(\lambda-\lambda_\sigma)^{g_\sigma}\eta_i^{(0)}(\lambda)$, the $\eta_i^{(0)}(\lambda)$ having been defined in section 2. The coefficients of t^k are derived from equations (11) after choosing g_σ so that the coefficients of t^k vanish for $k < d_{1\sigma}$, while for $k = d_{1\sigma}$ at least one coefficient of t^k in $P_{i\sigma}(t:\lambda)$ $(i=1,\cdots,n)$ is different from zero. The value of g_σ depends upon the vanishing or non-vanishing of the $D_i(\lambda_\sigma + d_{j\sigma})$ $(i=1,\cdots,n;\ j=1,\cdots,\sigma-1)$, and will be given explicitly in the subsequent discussion.

In considering the solution associated with $\lambda=\lambda_2$ it is seen that this solution will be of the form

$$x_i = t^{\lambda_2}P_{i2}(t) \qquad (i=1,\cdots,n),$$

if and only if

$$D_i(\lambda_2 + d_{12}) = 0 \qquad (i=1,\cdots,n). \tag{21}$$

Conditions (21) will in general not be satisfied, in which case the solution in question, just as in Case II of section 2, will be of the form

$$x_i = t^{\lambda_2}[(\partial P_{i2}(t:\lambda)/\partial\lambda) + P_{i2}(t:\lambda)\log t]_{\lambda=\lambda_2} \quad (i=1,\cdots,n).$$

If we assume for any one of the group of e roots of $D(\lambda)=0$, say for $\lambda=\lambda_\sigma$, that

$$D_i(\lambda_\sigma + d_{j\sigma}) \neq 0 \tag{22}$$

for at least one of the values $i=1,\cdots,n$ when $j=1,2,\cdots,\sigma-1$ respectively, then corresponding to $\lambda=\lambda_\sigma$, one readily verifies the following solution of system (1):

$$x_i = t^{\lambda\sigma}\left\{\sum_{h=0}^{\sigma-1}\binom{\sigma-1}{h}\frac{\partial^{\sigma-h-1}P_{i\sigma}(t:\lambda)}{\partial\lambda^{\sigma-h-1}}\log^h t\right\}_{\lambda=\lambda\sigma} \qquad (i=1,\cdots,n), \tag{23}$$

Clearly $g_\sigma = \sigma-1$ in the $P_{i\sigma}(t:\lambda)$ of solution (23). On substituting (23) in (1) it is observed that the coefficient of $\log^h t$ vanishes for $h=1,2,\cdots,\sigma-1$ respectively. The vanishing of the coefficients of the respective powers of $\log t$ is a consequence of the consistency of pairs of equation-sets of precisely the forms given by (11) and (12).

If it should happen for any root, $\lambda=\lambda_\epsilon$, of the group that

$$D_i(\lambda_\epsilon + d_{j\epsilon}) = 0 \qquad (i=1,\cdots,n)$$

for $j=1,2,\cdots,\epsilon-1$ respectively, then equations (11) will be consistent

when $\lambda = \lambda_\epsilon$ for all values of p. The corresponding solution of system (1) is therefore of the form

$$x_i = t^{\lambda\epsilon} P_{i\epsilon}(t) \qquad (i = 1, \cdots, n),$$

in which the $P_{i\epsilon}(t)$ $(i = 1, \cdots, n)$ are power series in t, at least one of which has a non-vanishing constant term. It is observed that the factor $(\lambda - \lambda_\sigma)^{g_\sigma}$ is introduced in the constant terms of the $P_{i\sigma}(t:\lambda)$ for the purpose of forming consistent sets of equations (12) when p takes on values $d_{j\sigma}$ for which the determinant $D(\lambda_\sigma + d_{j\sigma}) = 0$. Thus g_σ is reduced by one for each value of j for which

$$(24) \qquad D_i(\lambda_\sigma + d_{j\sigma}) = 0 \qquad (i = 1, \cdots, n); \ (j = 1, 2, \cdots, \sigma - 1).$$

One can now generalize solution (23) and write for any root $\lambda = \lambda_\sigma$ of $D(\lambda) = 0$

$$(25) \qquad x_i = t^{\lambda\sigma} \left\{ \sum_{h=0}^{g} \binom{g}{h} \left(\frac{\partial^{g-h} P_{i\sigma}(t:\lambda)}{\partial \lambda^{g-h}} \right) \log^h t \right\}_{\lambda=\lambda_\sigma} \qquad (i = 1, \cdots, n),$$

where, in view of the foregoing discussion,

$$g = \sigma - 1 - \rho,$$

ρ being the number of times (24) holds for the values $j = 1, 2, \cdots, \sigma - 1$.

Before treating the most general situation that can arise, namely, the case for which one or more of the $\lambda_1, \cdots, \lambda_\sigma$ are multiple roots, we shall take up in some detail the method of obtaining the solutions corresponding to a multiple root which is not congruent (mod 1) to any other root of $D(\lambda) = 0$.

4. *Solutions corresponding to a Multiple Root of* $D(\lambda) = 0$. Let $\lambda = \lambda_\sigma$ be a multiple root of $D(\lambda) = 0$ such that λ_σ does not differ from any other root of $D(\lambda) = 0$ by an integer. The nature of the solutions of system (1) for the present case depend upon the properties of the determinant $D(\lambda_\sigma)$ and will be obtained from hypotheses regarding $D(\lambda_\sigma)$ and its minors, that include every special case that can arise.

Suppose (A) that $\lambda = \lambda_\sigma$ is a root of $D(\lambda) = 0$ of multiplicity s_0 and (B) that all 1st, 2nd, $\cdots$, $(q-1)$st minors of $D(\lambda_\sigma)$ are zero, $q \leqq s_0$, while at least one qth minor of $D(\lambda_\sigma)$ is different from zero.

In consequence of hypotheses (A) and (B) equations (10) will have q linearly independent solutions each of which gives rise to q solutions of (1) of the form

$$x_i = t^{\lambda\sigma} P_{ikq}(t) \qquad (i = 1, \cdots, n; \ k = 1, \cdots, q),$$

where the $P_{ikq}(t)$ are power series in t of the form

$$P_{ikq}(t) = \sum_{h=0}^{\infty} a^{(h)}{}_{ikq}\ t^h \qquad (i = 1, \cdots, n).$$

The method of obtaining the $a^{(h)}{}_{ikq}$ has already been given in section 1.

The remaining $s_0 - q$ solutions, as will be shown, occur in groups of the form

$$x_i = t^{\lambda\sigma} \left\{ \sum_{h=0}^{k} \frac{\partial^{k-h} P_i(t:\lambda)}{\partial\lambda^{k-h}} \frac{\log^h t}{(r+k-h)!\,h!} \right\}_{\lambda=\lambda\sigma} \tag{26}$$
$$(i = 1, \cdots, n;\ k = 1, \cdots, \omega),$$

where

$$P_i(t:\lambda) = \sum_{h=0}^{\infty} a_i{}^{(h)}(\lambda) t^h.$$

The integers r and ω, and the coefficients $a_i{}^{(h)}(\lambda)$ will now be determined.

On substituting (26) in (1) it is seen that (26) will be a solution of system (1) for $k = 1, 2, \cdots, \omega$ provided the $a_i{}^{(0)}(\lambda)$ can be determined so as to satisfy the following conditions:

$$(\theta_{ii}{}^{(0)} - \lambda) a_i{}^{(0)}(\lambda) + \sum_{k=1}^{n} \theta_{ik}{}^{(0)} a_k{}^{(0)}(\lambda) = 0 \tag{27}$$
$$(i = 1, \cdots, n;\ k \neq i;\ \lambda = \lambda_\sigma),$$

$$(\theta_{ii}{}^{(0)} - \lambda) \frac{\partial^m a_i{}^{(0)}(\lambda)}{\partial\lambda^m} + \sum_{k=1}^{n} \theta_{ik}{}^{(0)} \frac{\partial^m a_k{}^{(0)}(\lambda)}{\partial\lambda^m} = (r+m) \frac{\partial^{m-1} a_i{}^{(0)}(\lambda)}{\partial\lambda^{m-1}} \tag{28}$$
$$(i = 1, \cdots, n;\ k \neq i;\ \lambda = \lambda_\sigma;\ m = 1, \cdots, \omega).$$

It will now be shown that under certain conditions equations (27) and (28) will be consistent and that the sufficient conditions for their consistency are implied in the hypotheses of the present case.

From the determinant $D(\lambda)$ one verifies the identities,

$$\begin{cases} (\theta_{ii}{}^{(0)} - \lambda)\delta_{ij}(\lambda) + \sum_{k=1}^{n} \theta_{ik}{}^{(0)} \delta_{kj}(\lambda) \overset{\lambda}{\equiv} 0 \\ \qquad (i = 1, \cdots, n-j;\ k \neq i), \\ (\theta_{ii}{}^{(0)} - \lambda)\delta_{ij}(\lambda) + \sum_{k=1}^{n} \theta_{ik}{}^{(0)} \delta_{kj}(\lambda) \overset{\lambda}{\equiv} \delta_{i,j-1}(\lambda) \\ \qquad (i = n-j+1, \cdots, n;\ k \neq i), \end{cases} \tag{29}$$

where the $\delta_{ij}(\lambda)$ $(i = 1, \cdots, n-j+1)$ are the jth minor-cofactors respectively (of order $n-j$), of the elements $\theta^{(0)}{}_{n-j+1,i}$ $(i = 1, \cdots, n-j+1)$, in the upper left $(j-1)$st minor of $D(\lambda)$; and the $\delta_{i,j-1}(\lambda)$, $(i = n-j+1, \cdots, n)$ are the upper left $(j-1)$st minors of $D(\lambda)$ obtained by replacing the $(n-j+1)$st row by the $(n-j+1)$st, $(n-j+2)$nd, $\cdots$, nth

rows respectively. For the foregoing particular choice in the position of the $(j-1)$st minor-cofactors occurring in (29) the $\delta_{kj}(\lambda)$, $(k = n-j+2, \cdots, n$ are identically zero, by definition, and have been written for the purpose of securing generality in notation and logic, and also for the purpose of making the coefficients in (29) agree exactly, in form, with the coefficients in the left numbers of equations (27) and (28). This specialization in the position, in $D(\lambda)$, of the $(j-1)$st minors in question has been employed solely for the purpose of effecting simplicity in presentation; all subsequent work, however, is entirely general and applies to any minor-cofactors satisfying certain conditions which will now be specified.

Suppose (a) that $(\lambda-\lambda_\sigma)^s$ is a factor of every $(j-1)$st minor (of order $n-j+1$) of $D(\lambda)$, while at least one $(j-1)$st minor does not contain the factor $(\lambda-\lambda_\sigma)^{s+1}$; and (b) that $(\lambda-\lambda_\sigma)^r$ is a factor of every jth minor in $D(\lambda)$, while at least one jth minor does not contain $(\lambda-\lambda_\sigma)^{r+1}$, where obviously $s-r \geqq 1$. Let identities (29) be differentiated $s-1$ times successively with respect to λ, setting $\lambda = \lambda_\sigma$ in each of the s sets of relations thus obtained. By hypothesis (a), the right members of the resulting s sets of relations vanish and, by (b), each term of the left members of the first r sets of equations also vanishes. There remain, however, the following $s-r$ non-trivial sets of relations:

$$(30)\qquad \begin{cases} (\theta_{ii}^{(0)}-\lambda_\sigma)\ \dfrac{\partial^r\delta_{ij}}{\partial\lambda^r}+\sum\limits_{k=1}^{n}\theta_{ik}^{(0)}\ \dfrac{\partial^r\delta_{kj}}{\partial\lambda^r}=r\,\dfrac{\partial^{r-1}\delta_{ij}}{\partial\lambda^{r-1}}=0 \\ \qquad\qquad (i=1,\cdots,n;\ k\neq i) \\ (\theta_{ii}^{(0)}-\lambda_\sigma)\ \dfrac{\partial^{r+h}\delta_{ij}}{\partial\lambda^{r+h}}+\sum\limits_{k=1}^{n}\theta_{ik}^{(0)}\ \dfrac{\partial^{r+h}\delta_{kj}}{\partial\lambda^{r+h}}=(r+h)\,\dfrac{\partial^{r+h-1}\delta_{ij}}{\partial\lambda^{r+h-1}} \\ \qquad\qquad (i=1,\cdots,n;\ k\neq i;\ h=1,\cdots,s-r-1), \end{cases}$$

where

$$\frac{\partial^\nu\delta_{ij}}{\partial\lambda^\nu} \equiv \left[\frac{\partial^\nu\delta_{ij}(\lambda)}{\partial\lambda^\nu}\right]_{\lambda=\lambda_\sigma}:$$

On comparing relations (30) with equations (27) and (28), it is seen that, under hypotheses (a) and (b), the latter equations have as solutions

$$(31)\qquad \begin{cases} a_i^{(0)}(\lambda) = \dfrac{\partial^r\delta_{ij}(\lambda)}{\partial\lambda^r} \qquad (i=1,\cdots,n;\ \lambda=\lambda_\sigma) \\ \dfrac{\partial^m a_i^{(0)}(\lambda)}{\partial\lambda^m} = \dfrac{\partial^{r+m}\delta_{ij}(\lambda)}{\partial\lambda^{r+m}} \\ \qquad\qquad (i=1,\cdots,n;\ m=1,\cdots,\ s-r-1;\ \lambda=\lambda_\sigma). \end{cases}$$

in which the value of r is defined by (b) and where $\omega = s - r - 1$. Thus from assumptions (a) and (b) one obtains $\omega = s - r - 1$ solutions of system (1) which are given by (26), for $k = 1, 2, \cdots, s - r - 1$.

In consequence of hypotheses (B) of the present section all derivatives with respect to λ, up to and including the $(q - j - 1)$st, of all jth minors of $D(\lambda)$ vanish when $\lambda = \lambda_\sigma$ for $j = 1, 2, \cdots, (q - 1)$. This follows from the fact that the λ-derivative of every $(j - 1)$st minor of $D(\lambda)$ is a sum of jth minors of $D(\lambda)$. Hence (B) implies that $(\lambda - \lambda_\sigma)^{q-j}$ is a factor of every jth minor of $D(\lambda)$, $j = 1, \cdots, q - 1$. If there exists at least one first minor of $D(\lambda)$ which does not contain $(\lambda - \lambda_\sigma)^q$ as a factor, then in view of the foregoing discussion the remaining $s_0 - q$ solutions of system (1), associated with $\lambda = \lambda_\sigma$, are given by (26) with

$$s = s_0, \quad r = q - 1, \quad j = 1, \quad \omega = s_0 - q.$$

It may happen, however, that $(\lambda - \lambda_\sigma)^\alpha$ where $\alpha \geqq q$ is a common factor of all first minors of $D(\lambda)$ in which case there are fewer than $s_0 - q$ solutions of system (1) which are expressed in terms of first minor-cofactors of $D(\lambda)$ and their λ-derivatives. In fact a similar special situation may exist for the jth minors of $D(\lambda)$ for $j = 2, \cdots, q - 1$. The assumptions with respect to this question will therefore be generalized so as to include every possible special case.

Hence let it be assumed (C) that every jth minor of $D(\lambda)$, has the factor $(\lambda - \lambda_\sigma)^{s_j}$ while at least one jth minor of $D(\lambda)$ does not contain $(\lambda - \lambda_\sigma)^{s_j+1}$, $(j = 1, \cdots, q)$, where obviously $s_j \geqq q - j$ and $s_{j-1} - s_j \geqq 1$. Hypotheses (C) are identical in character with the assumptions (a) and (b) underlying the derivation of (31), and consequently form (26) gives solutions of system (1) in which the coefficients of the respective powers of t are expressed in terms of minor-cofactors in $D(\lambda_\sigma)$ and their λ-derivatives. In obtaining solutions of system (1) for which the $a_i^{(0)}$ are expressed in terms of jth minor-cofactors of $D(\lambda_\sigma)$, $(j = 1, \cdots, q)$ one makes the following substitutions in (26) and (31):

$$s = s_{j-1}; \quad r = s_j; \quad \omega = s_{j-1} - s_j - 1 \equiv \omega_j. \tag{32}$$

The total number of solutions, corresponding to $\lambda = \lambda_\sigma$, which one obtains in this way is clearly given by

$$\begin{aligned} \omega_1 + \omega_2 + \cdots + \omega_q &= \\ &= (s_0 - s_1 - 1) + (s_1 - s_2 - 1) + \cdots + (s_{q-1} - s_q - 1) \\ &= s_0 - s_q - q = s_0 - q, \end{aligned}$$

since, by (B), $s_q = 0$.

Thus we have obtained s_0 solutions of equations (1) associated with $\lambda = \lambda_\sigma$ which is a root of $D(\lambda) = 0$ of order s_0. These solutions may be conveniently represented by the notation

$$(33) \qquad x_i = \sum_{k=1}^{q} c_{\sigma k}\, t^{\lambda_\sigma} P_{ikQ}(t) + \sum_{j=1}^{q} \sum_{\omega=1}^{\omega_j} c_{\sigma\omega j}\, t^{\lambda_\sigma} R_{i\omega j}(t) \quad (i = 1, \cdots, n),$$

where the $t^{\lambda_\sigma} R_{i\omega j}(t), \omega = 1, \cdots, \omega_j$ are the ω_j solutions given by (26) in terms of jth minor-cofactors of $D(\lambda_\sigma)$ and their λ-derivatives.

5. *Solutions Associated with a Group of Multiple Roots. General Case.* Let $\lambda_1, \cdots, \lambda_e$ be a group of multiple roots of $D(\lambda) = 0$ of orders $\beta_1, \cdots, \beta_e$ respectively, such that

$$\lambda_i - \lambda_j = d_{ij} \qquad (i, j = 1, \cdots, n;\ i < j),$$

where the d_{ij} are positive integers. It is also assumed that no other root of $D(\lambda) = 0$ differs from any one of the λ_j $(j = 1, \cdots, e)$ by an integer.

The results of sections 3and 4 enable one to obtain, without difficulty, the $\beta_1 + \cdots + \beta_e$ solutions of system (1) corresponding to the foregoing group of multiple roots. Consider λ_σ', any one of the e multiple roots, and assume that $D(\lambda_\sigma)$ has properties (A), (B), and (C) of section 4. Set $\beta_\sigma' = s_0$, in order that the results of section 4 may be applied with as little change in notation as possible.

On referring to equations (11) and taking $p = d_{k\sigma}$ $(k = 1, \cdots, \sigma - 1)$ it is noted that $D(\lambda + d_{k\sigma})$ contains the factor $(\lambda - \lambda_\sigma')^{\beta_k}$ but does not contain the factor $(\lambda - \lambda_\sigma')^{\beta_k + 1}$. It may happen that some power of $\lambda - \lambda_\sigma$ is a common factor of the n determinants $D_i(\lambda + d_{k\sigma})$, $(i = 1, \cdots, n)$. We shall therefore assume that $(\lambda - \lambda_\sigma')^{\gamma_k}$ is a factor of every $D_i(\lambda + d_{k\sigma})$, $(i = 1, \cdots, n)$ while at least one of these n determinants does not contain $(\lambda - \lambda_\sigma')^{\gamma_k + 1}$. The foregoing hypotheses apply for $k = 1, 2, \cdots, \sigma - 1$. Let

$$(34) \qquad (\beta_1 + \beta_2 + \cdots + \beta_{\sigma-1}) - (\gamma_1 + \gamma_2 + \cdots + \gamma_{\sigma-1}) = g(\sigma; \delta_i(\lambda)) \equiv g,$$

and choose

$$(35) \qquad a_i^{(0)}(\lambda) = (\lambda - \lambda_\sigma)^g\, \delta_i(\lambda) \qquad (i = 1, \cdots, n).$$

In view of the hypotheses (A) and (B), equations (10), for $\lambda = \lambda_\sigma$, have q linearly independent sets of solutions which are expressed in terms of qth minor-cofactors of $D(\lambda_\sigma)$. Replace the λ_σ' by λ in these solutions and thus obtain q sets of functions $\delta_{iqk}(\lambda)$.

Set $\delta_i(\lambda) = \delta_{iqk}(\lambda)$, $(k = 1, \cdots, q)$ respectively, in (35), giving q sets of values of the $a_i^{(0)}(\lambda)$. Equations (11) now determine the $a_i^{(p)}(\lambda)$

$(p = 1, 2, \cdots)$ as functions of λ. Substitute the q sets of values for the coefficients $a_i^{(p)}$ in the series

$$P_{ikq}(t:\lambda) = \sum_{h=0}^{\infty} a_i^{(h)}\, t^h \qquad (i = 1, \cdots, n),$$

and indicate the resulting series by the notation

$$P_{ikq}(t;\lambda) = \sum_{h=0}^{\infty} a_{ikq}^{(h)}(\lambda)\, t^h \qquad (i = 1, \cdots, n;\ k = 1, \cdots, q).$$

Since, in general, the value of g depends upon the choice of the $\delta_i(\lambda)$ we shall designate by α_k the value of g corresponding to $\delta_i(\lambda) = \delta_{ikq}(\lambda)$. One can now write q of the desired s_0 solutions of system (1) in the form

$$(36) \qquad x_i = t^{\lambda\sigma} \left\{ \sum_{h=0}^{\alpha_k} \frac{\partial^{\alpha_k - h} P_{ikq}(t:\lambda)}{\partial\lambda^{\alpha_k - h}} \frac{\log^h t}{(\alpha_k - h)!\, h!} \right\}_{\lambda=\lambda\sigma} \qquad (i = 1, \cdots, n;\ k = 1, \cdots, q).$$

In order to obtain the remaining $s_0 - q$ solutions of system (1) choose for the $\delta_i(\lambda)$ in (35) exactly the λ-derivatives of jth minor-cofactors of $D(\lambda)$, $(j = 1, \cdots, q)$ given in (31), which were used in setting up the $R_{i\omega j}(t)$ of (33). Designate by μ_j the value of g resulting from the choice

$$\delta_i(\lambda) = \frac{\partial^{s_j}\, \delta_{ij}(\lambda)}{\partial\lambda^{s_j}} .$$

Let the corresponding sets of infinite series in t be represented by $R_{ij}(t:\lambda)$ $(i = 1, \cdots, n)$ and set

$$(37) \qquad \sum_{h=0}^{\mu_j} \frac{\partial^{\mu_j - h} R_{ij}(t:\lambda)}{\partial\lambda^{\mu_j - h}} \frac{\log^{h+k} t}{(h+k)!\,(\mu_j - h)!} = H_{ijk}(t:\lambda) \quad (i = 1, \cdots, n).$$

We shall find that the remaining $s_0 - q$ solutions of system (1) corresponding to $\lambda = \lambda_\sigma$ can be expressed in the form

$$(38) \qquad x_i = t^{\lambda\sigma} \left\{ \frac{H_{ijk}(t:\lambda)}{s_j!} + \sum_{\nu=1}^{k} \frac{\partial^{\mu_j + \nu} R_{ij}(t:\lambda)}{\partial\lambda^{\mu_j + \nu}} \frac{\log^{k-\nu} t}{(k-\nu)!\,(s_j + \nu)!} \right\}_{\lambda=\lambda\sigma}$$
$$(i = 1, \cdots, n;\ k = 1, \cdots, \omega_j;\ j = 1, \cdots, q).$$

The hypotheses regarding $D(\lambda_\sigma)$ and its minor-cofactors are identical with those underlying the derivation of solutions (33) and hence the results of section 4 show that (38) gives $s_0 - q$ solutions of system (1).

If the x_i of (38) be substituted in (1) the coefficients of $\log^m t$ $(m = 0, 1, \cdots, \mu_j + \omega_j - 1)$ vanish in each of the resulting n equations.

For each of the foregoing values of m, the coefficient of $\log^m t$ when equated to zero, gives identities of exactly the forms (11) and (12); while for $m = \mu_j + \omega_j$ the coefficient of $\log^m t$ vanishes in consequence of identities of the form (10).

6. *Independence of the n Solutions of System* (1). Let

$$x_i = \phi_{ij}(t) \qquad (i = 1, \cdots, n;\ j = 1, \cdots, n) \tag{39}$$

be a set of n solutions of system (1) obtained by the foregoing method. These solutions will form a fundamental set if and only if their determinant

$$\Delta(t) = | \phi_{ij}(t) | \not\equiv 0.$$

The n solutions given by (39) occur in groups of the form

$$x_i = t^{\lambda\sigma} \{ \sum_{k=1}^{l_{\sigma j}} V_{ij\sigma}^{(k-1)}(t) \log^{k-1} t \} \tag{40}$$
$$(j = 1, \cdots, q_\sigma;\ \sigma = 1, \cdots, n;\ i = 1, \cdots, n)$$

where $$1 \leqq l_{\sigma j} \leqq n \quad \text{and} \quad \sum_{\sigma=1}^{n} \sum_{j=1}^{q_\sigma} l_{\sigma j} = n.$$

Thus $\Delta(t)$ may be reduced to the form

$$\Delta(t) = t^N | V_{ij\sigma}^{(0)}(t) | = t^N K(t) \tag{41}$$
$$(j = 1, \cdots, q_\sigma;\ \sigma = 1, \cdots, n;\ i = 1, \cdots, n),$$

where the $V_{ij\sigma}^{(0)}(t)$ are the power series in t, that are not multiplied by a power of $\log t$, in each of the n solutions; and N is the sum of the n roots of $D(\lambda) = 0$.

Clearly $\Delta(t) \overset{t}{\equiv} 0$ if and only if $K(t) \overset{t}{\equiv} 0$. Since $K(t)$ is an analytic function of t in the neighborhood of $t = 0$, $K(0) \not\equiv 0$ implies that $\Delta(t_1) \not\equiv 0$ for $| t_1 |$ sufficiently small. Thus the independence of the n solutions of system (1) is established if it can be shown that $K(0) \not\equiv 0$.

If we write

$$V_{ij\sigma}^{(0)}(t) = a_{ij\sigma}^{(0)} + a_{ij\sigma}^{(1)} t + a_{ij\sigma}^{(2)} t^2 + \cdots$$
$$(j = 1, \cdots, q_\sigma;\ \sigma = 1, \cdots, n;\ i = 1, \cdots, n),$$

it is observed that

$$K(0) = | a_{ij\sigma}^{(0)} | \qquad (j = 1, \cdots, q_\sigma;\ \sigma = 1, \cdots, n;\ i = 1, \cdots, n), \tag{42}$$

where $a_{ij\sigma}^{(0)} = 0$, by definition, if there are no $V_{ij\sigma}^{(0)}(t)$ in which the $a_{ij\sigma}^{(0)}$ are expressed in terms of jth minor-cofactors of $D(\lambda_\sigma)$.

Now consider the system of differential equations

$$dx_i/dt = \theta_{i1}^{(0)} x_1 + \cdots + \theta_{in}^{(0)} x_n \qquad (i = 1, \cdots, n), \tag{43}$$

obtained from system (1) by omitting the factor t from the left members and retaining as coefficients of the $x_1, \cdots, x_n$ the constant terms of the $\theta_{i1}(t), \cdots, \theta_{in}(t)$. The n solutions of system (43) as obtained by a direct method due to the author,* are shown to form a fundamental set. The proof of this property of the n solutions, in the paper referred to, is effected by showing that the determinant $E(t) = E(0)$

$$\exp \int \sum \theta_{ii}^{(0)} dt$$

of the solutions is different from zero for $t = 0$. The determinant $E(0)$ is identical with the determinant $K(0)$ of the present paper, except for a constant non-zero multiplier.

Hence the independence of the n solutions of system (1) follows as a consequence of the corresponding proof for the solutions of a system of linear differential equations having constant coefficients.

7. *Convergence of the Solutions.* Let it be assumed that the $\theta_{ij}(t)$ of system (1) converge for $|t| \leqq R$. Choose $t = t_0$ any point within the domain of convergence. Let

$$x_i = \phi_{ij}(t) \qquad (i, j = 1, \cdots, n), \tag{44}$$

be a fundamental set of solutions for the regular point $t = t_0$. Make analytic continuations of the ϕ_{ij} about $t = 0$ along paths which neither enclose nor pass through any other singular point.

Let $\phi_{ij}(t)$ become $\bar{\phi}_{ij}(t)$ after making this circuit. Since the $\bar{\phi}_{ij}$ that result from this circuit are also solutions of system (1), one can write

$$\bar{\phi}_{ij}(t) = \sum_{k=1}^{n} c_{kj}\, \phi_{ik}(t) \qquad (i = 1, \cdots, n) \tag{45}$$

for $j = 1, \cdots, n$.

We shall raise the question of the existence of a solution of system (1),

$$x_i = \phi_i(t) \qquad (i = 1, \cdots, n),$$

having the property

$$\bar{\phi}_i(t) = r\, \phi_i(t) \qquad (i = 1, \cdots, n). \tag{46}$$

If such a solution exists it can be written in the form

$$\phi_i(t) = \sum_{k=1}^{n} b_k\, \phi_{ik}(t) \qquad (i = 1, \cdots, n), \tag{47}$$

where, after a circuit about $t = 0$ one has

$$\bar{\phi}_i(t) = \sum_{k=1}^{n} b_k\, \bar{\phi}_{ik}(t). \tag{48}$$

* *American Journal of Mathematics*, Vol. 47 (1925), pp. 257-276.

After expressing both members of (46) as linear functions of the $\phi_{ij}(t)$ by substituting from (45), (47), and (48), there results

$$r\sum_{k=1}^{n} b_k\,\phi_{ik} = \sum_{k=1}^{n} b_k\left(\sum_{j=1}^{n} c_{kj}\,\phi_{ij}(t)\right) \qquad (i=1,\cdots,n). \tag{49}$$

Relations (49) may be regarded as a system of linear homogeneous equations in the b_i $(i=1,\cdots,n)$; and they will have a solution other than $b_i=0$ $(i=1,\cdots,n)$ if and only if the determinant of the coefficients

$$\Gamma(r)\Delta(r) \equiv \begin{vmatrix} c_{11}-r, & c_{12}, & \cdots, & c_{1n} \\ c_{21}, & c_{22}-r, & \cdots, & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n1}, & c_{n2}, & \cdots, & c_{nn}-r \end{vmatrix} \cdot \begin{vmatrix} \phi_{11}, & \phi_{12}, & \cdots, & \phi_{1n} \\ \phi_{21}, & \phi_{22}, & \cdots, & \phi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \phi_{n1}, & \phi_{n2}, & \cdots, & \phi_{nn} \end{vmatrix} = 0. \tag{50}$$

Since, by hypothesis, (44) is a fundamental set of solutions of system (1) it is evident that (50) is equivalent to $\Gamma(r)=0$. Thus for each root of $\Gamma(r)=0$ there exists a set of values of the b_i given by (49) which upon substitution in (47) yield the solution $x_i=\phi_i(t)$, satisfying condition (46).

Suppose first that the n roots $r_1,\cdots,r_n$ of $\Gamma(r)=0$ are distinct and that the corresponding n sets of solutions of system (1) having property (46) are given by

$$x_i = \psi_{ij}(t) \qquad (i,j=1,\cdots,n), \tag{51}$$

where

$$\overline{\psi}_{ij}(t) = r_j\,\psi_{ij}(t) \qquad (i=1,\cdots,n). \tag{52}$$

Let r_σ be any root of $\Gamma(r)=0$ and define λ_σ by the relation

$$r_\sigma = e^{2\pi i\lambda_\sigma}, \tag{53}$$

from which one has

$$\lambda_\sigma = (1/2\pi i)\log r_\sigma + h, \tag{54}$$

where h is any integer. Consider the function

$$U_\sigma(t) = t^{\lambda_\sigma}$$

which, after a circuit about the point $t=0$ becomes

$$\overline{U}_\sigma(t) = r_\sigma U_\sigma(t). \tag{55}$$

Thus one observes that the ratio

$$\psi_{ij}(t)/U_j(t) \equiv P_{ij}(t) \qquad (i,j=1,\cdots,n),$$

is a uniform function in the neighborhood of $t = 0$. Hence one has, in view of the treatment of section 1

$$x_i = \psi_{ij}(t) = t^{\lambda_j} P_{ij}(t) = t^{\lambda_j} \sum_{k=0}^{\infty} a_{ij}^{(k)} t^k \tag{56}$$
$$(i = 1, \cdots, n;\ j = 1, \cdots, n),$$

where $\lambda = \lambda_j$ is a root of the characteristic equation $D(\lambda) = 0$. As a consequence of the unique existence of the solutions of analytic differential equations it follows that the $P_{ij}(t)$ of (56) converge for any value of t for which $|t| < R$.

If $r = r_\sigma$ is a root of $\Gamma(r) = 0$ of multiplicity μ then, by the foregoing argument, one proves the existence of at least one solution of the form (56). All solutions of system (1), corresponding to $r = r_\sigma$ and not of the form (56), will be shown to occur in groups, and be of the form

$$x_i = t^{\lambda_\sigma} \{P_{i0}(t) + P_{i1}(t) \log t + \cdots + P_{ik}(t) \log^k t\}$$
$$(i = 1, \cdots, n).$$

We shall start with a fundamental set of solutions of system (1), for the regular point $t = t_0$ and first obtain the conditions under which it is possible to determine solutions

$$x_i = \psi_{ij}(t) = \sum_{j=1}^{n} b_j^{(k)} \phi_{ij}(t) \tag{57}$$
$$(i = 1, \cdots, n;\ k = 0, 1, \cdots, \epsilon),$$

such that, after a circuit about $t = 0$ one has

$$\begin{cases} \bar{\psi}_{i0}(t) = r_\sigma \psi_{i0}(t) \\ \bar{\psi}_{i1}(t) = r_\sigma \psi_{i1}(t) + \psi_{i0}(t) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \bar{\psi}_{i\epsilon}(t) = r_\sigma \psi_{i\epsilon}(t) + \psi_{i,\epsilon-1}(t) \end{cases} \tag{58}$$
$$(i = 1, \cdots, n).$$

On substituting (57) and (45) in (58), it is not difficult to show that sufficient conditions for the existence of (58) may be written in the form

$$(c_{ii} - r_\sigma) b_i^{(0)} + \sum_{k=1}^{n} c_{ik} b_k^{(0)} = 0 \qquad (i = 1, \cdots, n;\ k \neq i), \tag{59}$$

$$(c_{ii} - r_\sigma) b_i^{(h)} + \sum_{k=1}^{n} c_{ik} b_k^{(h)} = b_i^{(h-1)} \tag{60}$$
$$(i = 1, \cdots, n;\ k \neq i;\ h = 1, \cdots, \epsilon).$$

A set of values of the $b_i^{(0)}$ not all of which are zero, will satisfy (59) provided r_σ is a root of the equation $\Gamma(r) = 0$. The values of the $b_i^{(h)}$ satisfying (60) have already been given in article 4. Thus we have shown

the existence of the solution of system (1) given in (57) and having the properties stated in (58).

Define L_1 and L_k $(k = 2, \cdots, \epsilon)$ by the relations

$$(61) \qquad \begin{cases} L_1 = (1/2\pi i) \log t \\ L_k = L_1(L_1 - 1) \cdots (L_1 - k + 1)/k! \,. \end{cases}$$

Indicate, by the notation $\overline{L}_k$, the value of L_k after t has described a circuit about $t = 0$. One easily verifies that

$$(62) \qquad \begin{cases} \overline{L}_1 = L_1 + 1 \\ \overline{L}_k = L_k + L_{k-1} \qquad\qquad (k = 2, \cdots, \epsilon). \end{cases}$$

Let the functions $Q_{i0}(t)$, $Q_{i1}(t), \cdots, Q_{i\epsilon}(t)$ be defined by the identities

$$(63) \qquad \begin{cases} \psi_{i0}(t) = t^{\lambda_\sigma} Q_{i0}(t), \\ \psi_{i1}(t) = t^{\lambda_\sigma} [Q_{i1}(t) + (1/r_\sigma) Q_{i0}(t)\, L_1], \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \psi_{i\epsilon}(t) = t^{\lambda_\sigma} [Q_{i\epsilon}(t) + (1/r_\sigma)\, Q_{i,\epsilon-1}(t) L_1 + \cdots + (1/r_\sigma{}^{\epsilon-1})\, Q_{i0}(t)\, L_{\epsilon-1}] \end{cases}$$

Designate by $\overline{Q}_{ij}(t)$ the value of $Q_{ij}(t)$ after a circuit about $t = 0$. On using relations (55), (58), and (62) one easily verifies, in (63), that

$$\overline{Q}_{ij}(t) = Q_{ij}(t) \qquad (i = 1, \cdots, n;\ j = 0, \cdots, \epsilon).$$

Thus it is seen that the functions $Q_{ij}(t)$ are uniform in the neighborhood of $t = 0$. From the method of setting up the solutions of system (1), employed in the present paper, it follows that the $Q_{ij}(t)$ are power series in t.

Hence it has been shown that in the case of a multiple root $r = r_\sigma$ of $\Gamma(r) = 0$, the solutions of system (1) at the regular point $t = t_0$ occur in groups, and can be written in the form

$$(64) \qquad x_i = \psi_{ij}(t) = t^{\lambda_\sigma} [P_{i0}(t) + P_{i1}(t) \log t + \cdots + P_{ih}(t) \log^h t] \qquad (i = 1, \cdots, n;\ h = 0, 1, \cdots, \epsilon),$$

where the $P_{ij}(t)$ are the power series in t that have already been determined in the foregoing sections. Since solutions (64) exist uniquely for every regular point it follows that the $P_{ij}(t)$ converge for $|t| < R$.

For any root $r = r_\sigma$ of $\Gamma(r) = 0$ it is observed that the corresponding λ_σ as given by (54) is determined only to an additive integer. Thus the above convergence proof for a multiple root $r = r_\sigma$ of $\Gamma(r) = 0$ includes the general case of article 5 in which the solutions of system (1) are obtained corresponding to a group of multiple roots of $D(\lambda) = 0$, mutually congruent (mod 1).

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On Related Difference and Differential Systems.*

By William M. Whyburn.

1. *Introduction.* Sturm's † work on difference equations has been reconstructed for the most part by Porter,‡ Bocher,§ Fort,¶ Merrill,‖ Carmichael,** and others. In addition numerous authors including Hilbert †† and Fredholm ‡‡ have made extensive use of the intimate relationship that exists between algebraic and transcendental problems. Porter ‡ was the first to actually carry difference equation results over to the limiting differential equations. Others have carried this limiting process through, but all previous results have been confined to the case where the coefficients of the differential systems are continuous functions. It is the purpose of the present paper to establish results for difference systems and to carry these results over to the limiting differential systems. This passage to the limit is accomplished for a differential system with Lebesgue summable coefficients by making use of a subdivision of the interval of the independent variable.

In parts 2 and 3, preliminary theorems concerning a system of two linear difference equations are proved. In part 4, we show that the equations, solutions, and many properties of a system of m difference equations of the first order go over in the limit to the equations, solutions, and properties, respectively of a system m differential equations with summable coefficients. Similar results are obtained for general non-linear systems. In part 5, the Green's function and the adjoint system are defined for difference systems and these quantities are carried over to the limiting differential systems.

The notation of vector and tensor calculus is used in parts 2, 4, and 5. A capital letter denotes an m-dimensional tensor (square matrix) whose element in the ith row and jth column is given by the same letter with the sub-

* Presented to the American Mathematical Society, September 9, 1926 and Dec. 31, 1926.

† *Journal des Mathématiques*, t. 1 (1836), pp. 106-186.

‡ *Annals of Mathematics*, Vol. 3 (1901-1902), pp. 55-70.

§ *Bulletin of the American Mathematical Society*, Vol. 18 (1911-1912), p. 1 ff.

¶ *American Journal of Mathematics*, Vol. 39 (1917), pp. 1-26.

‖ *Transactions of the American Mathematical Society*, Vol. 4 (1903), p. 423 ff.

** *American Journal of Mathematics*, Vol. 43 (1921), pp. 69-101, 232-270.

†† Six papers in *Göttinger Nachrichten*, 1904-1910.

‡‡ *Acta Mathematica*, Bd. 27, S. 384.

script ij and whose vector component that makes up its jth column is given by the same letter with the single subscript j. A lowercase letter or a Greek letter is used to denote an m-dimensional vector whose jth component is given by the same letter with the subscript j. To say that a vector or tensor has a certain property means that each of its components has that property. The dot product, $A \cdot B$, of the tensor A by the tensor B is a tensor C such that $C_{rj} = \sum_{k=1}^{m} A_{rk}B_{kj}$. The product $A \cdot u$ of the tensor A by the vector u is a vector b such that $b_j = \sum_{k=1}^{m} A_{jk}u_k$. The letters $j, k, i, m, n, p, r, s,$ are used to denote variables of iteration. All exceptions to the above notations are signalized where they occur.

Throughout the paper we will suppose that our difference systems are defined with reference to the finite set of distinct points $E_n : x_{0n} = a, x_{1n}, x_{2n}, \cdots, x_{nn} = b$ lying on the finite interval $X : a \leqq x \leqq b$. Bold-faced type denotes a function that is defined on E_n only. By $\boldsymbol{f}(i)$ we mean $\boldsymbol{f}(x_{in})$ and by $\Delta\boldsymbol{f}(i)$ we mean $\boldsymbol{f}(i+1) - \boldsymbol{f}(i)$. We use the notation $j = (k, r)$ to mean that j takes on the values k and r together with all positive integral values that lie between k and r.

2. *The Second Order System.* Consider the second order difference system

$$\frac{\Delta \boldsymbol{y}_j(i)}{\Delta x(i)} - \boldsymbol{A}_{j1}(i)\boldsymbol{y}_1(i) - \boldsymbol{A}_{j2}(i)\boldsymbol{y}_2(i) = 0 \qquad (j = 1, 2) \tag{2.1''}$$

which we can write in the form

$$\frac{\Delta \boldsymbol{y}(i)}{\Delta x(i)} - \boldsymbol{A}(i) \cdot \boldsymbol{y}(i) = 0 \tag{2.1'}$$

where $\boldsymbol{A}(i)$ is a two dimensional tensor and $\boldsymbol{y}(i)$ is a two dimensional vector. Let E denote the unit tensor which is defined so that $E_{ij} = 0$ for $i \neq j$ and $E_{jj} = 1$. Let $\boldsymbol{B}(i) = \boldsymbol{A}(i)\Delta x(i) + E$ and equation (2.1′) takes the form

$$\boldsymbol{y}(i+1) = \boldsymbol{B}(i) \cdot \boldsymbol{y}(i). \tag{2.1}$$

Throughout this discussion we will assume that $|\boldsymbol{B}(i)| \neq 0$ for every point of E_n, where $|\boldsymbol{B}(i)|$ means the determinant (second order) of $\boldsymbol{B}(i)$. We will frequently use continuous functions $f_n(x)$ that are defined in the following manner:

$$f_n(x) = \boldsymbol{f}(i) + \frac{\Delta \boldsymbol{f}(i)}{\Delta x(i)} [x - x_{in}] \quad \text{on } x_{in} \leqq x \leqq x_{i+1,n}.$$

The notation $f_n(x)$ will be used consistently to denote a function of this type that is constructed from $\boldsymbol{f}(i)$.

Definition: If $\boldsymbol{u}(i)$ and $\boldsymbol{v}(i)$ are any two solutions of (2.1), then by the wronskian of these solutions we mean $\boldsymbol{W}(i)$, where

$$\boldsymbol{W}(i) = \begin{vmatrix} \boldsymbol{u}_1(i) & \boldsymbol{v}_1(i) \\ \boldsymbol{u}_2(i) & \boldsymbol{v}_2(i) \end{vmatrix}.$$

THEOREM 2.1: *$\boldsymbol{W}(i)$ either vanishes identically or it does not vanish at all on E_n and if $|\boldsymbol{B}(i)| > 0$ on E_n, $\boldsymbol{W}(i)$ maintains the same sign on E_n.*

Proof: Substitute $\boldsymbol{u}(i)$ and $\boldsymbol{v}(i)$ into (2.1) and write the resulting equations in component form.

$$\text{(a)} \qquad \boldsymbol{u}_1(i+1) = \boldsymbol{B}_{11}(i)\boldsymbol{u}_1(i) + \boldsymbol{B}_{12}(i)\boldsymbol{u}_2(i)$$

$$\text{(b)} \qquad \boldsymbol{u}_2(i+1) = \boldsymbol{B}_{21}(i)\boldsymbol{u}_1(i) + \boldsymbol{B}_{22}(i)\boldsymbol{u}_2(i)$$

$$\text{(c)} \qquad \boldsymbol{v}_1(i+1) = \boldsymbol{B}_{11}(i)\boldsymbol{v}_1(i) + \boldsymbol{B}_{12}(i)\boldsymbol{v}_2(i)$$

$$\text{(d)} \qquad \boldsymbol{v}_2(i+1) = \boldsymbol{B}_{21}(i)\boldsymbol{v}_1(i) + \boldsymbol{B}_{22}(i)\boldsymbol{v}_2(i).$$

Multiply equation (c) by $\boldsymbol{u}_2(i)$ and subtract this from (a) after the latter has been multiplied through by $\boldsymbol{v}_2(i)$. Multiply (d) by $\boldsymbol{u}_1(i+1)$ and subtract this from (b) after multiplying (b) through by $\boldsymbol{v}_1(i+1)$. This gives

$$\begin{aligned} \boldsymbol{v}_2(i)\boldsymbol{u}_1(i+1) - \boldsymbol{u}_2(i)\boldsymbol{v}_1(i+1) &= \boldsymbol{B}_{11}(i)\boldsymbol{W}(i), \\ -\boldsymbol{W}(i+1) &= \boldsymbol{B}_{21}(i)[\boldsymbol{u}_1(i)\boldsymbol{v}_1(i+1) - \boldsymbol{v}_1(i)\boldsymbol{u}_1(i+1)] + \\ &\quad \boldsymbol{B}_{22}(i)[\boldsymbol{u}_2(i)\boldsymbol{v}_1(i+1) - \boldsymbol{v}_2(i)\boldsymbol{u}_1(i+1)]. \end{aligned}$$

Combining these equations and replacing $\boldsymbol{v}_1(i+1)$ and $\boldsymbol{u}_1(i+1)$ by their values as given by (a) and (c), we get

$$\boldsymbol{W}(i+1) = |\boldsymbol{B}(i)|\,\boldsymbol{W}(i).$$

From this equation we conclude immediately that $\boldsymbol{W}(i)$ either vanishes identically or does not vanish at all and if $|\boldsymbol{B}(i)| > 0$ on E_n, then $\boldsymbol{W}(i)$ maintains the same sign on E_n.

Porter * proved many properties of second order difference systems that carry over without difficulty to our system. He proved that a necessary and sufficient condition that two solutions be linearly independent is that $\boldsymbol{W}(i)$ be different from zero on E_n. An examination of the matrix of coefficients reveals that the system (2.1) has a two parameter family of solutions since this matrix contains one determinant of the $2n$th order whose value is $\prod_0^{n-1} |\boldsymbol{B}(i)| \neq 0$. It follows from Porter's work that if $\boldsymbol{u}(i)$ and $\boldsymbol{v}(i)$ are linearly independent solutions of (2.1), then $\boldsymbol{u}_1(i)$ and $\boldsymbol{v}_1(i)$ cannot both

* *Loc. cit.*, pp. 59-60.

vanish for any point of E_n. It is also true that $\boldsymbol{v}_1(i)$ and $\boldsymbol{v}_2(i)$ cannot both vanish at the same point of E_n and, $\boldsymbol{v}_1(i)$ and $\boldsymbol{v}_1(i+1)$ cannot both vanish for any value of i.

THEOREM 2.2: *If $\boldsymbol{u}(i)$ and $\boldsymbol{v}(i)$ are two linearly independent solutions of (2.1) and $\varphi(i) = \boldsymbol{\alpha}(i) \cdot \boldsymbol{u}(i)$, $\theta(i) = \boldsymbol{\alpha}(i) \cdot \boldsymbol{v}(i)$, where $\alpha_1^2(i) + \alpha_2^2(i) \neq 0$, then the zeros of $\phi_n(x)$ and $\theta_n(x)$ separate each other on X provided $|\boldsymbol{B}(i)| > 0$ and $\{\alpha_1\alpha_2\}$ is different from zero and maintains the same sign on E_n, where*

$$\{\alpha_1\alpha_2\} = \alpha_2(i)\alpha_2(i+1)\boldsymbol{B}_{21}(i) + \alpha_1(i+1)\alpha_2(i)\boldsymbol{B}_{11}(i) - \alpha_1(i)\alpha_2(i+1)\boldsymbol{B}_{22}(i) - \alpha_1(i)\alpha_1(i+1)\boldsymbol{B}_{12}(i).$$

Proof: Neither $\phi_n(x)$ nor $\theta_n(x)$ can have more than a finite number of zeros on X. For, if we suppose that $\phi_n(x)$ has an infinite number of zeros on X, then there must be at least one pair of consecutive values of $\varphi(i)$ that are both zero. Let $\varphi(r) = 0$ and $\varphi(r+1) = 0$, then

$$\boldsymbol{\alpha}(r) \cdot \boldsymbol{u}(r) = 0$$
$$\boldsymbol{\alpha}(r+1) \cdot \boldsymbol{B}(r) \cdot \boldsymbol{u}(r) = 0.$$

The determinant of coefficients of $\boldsymbol{u}_1(r)$ and $\boldsymbol{u}_2(r)$ is $\{\alpha_1\alpha_2\}$ which is different from zero. This leads to a contradiction when we note that $\boldsymbol{u}_1(r)$ and $\boldsymbol{u}_2(r)$ cannot both vanish.

$\phi_n(x)$ and $\theta_n(x)$ cannot both vanish at the same point of X. Suppose that at some point t on $x_{rn} \leqq t \leqq x_{r+1,n}$ we have $\phi_n(t) = 0$ and $\theta_n(t) = 0$. It follows from the statement that we have just proved that $\Delta\varphi(r) \neq 0$, $\Delta\theta(r) \neq 0$ and hence we can solve the equations $\phi_n(t) = 0$, $\theta_n(t) = 0$ for t.

$$t = x_{rn} - \frac{\varphi(r)\Delta x(r)}{\Delta\varphi(r)} = x_{rn} - \frac{\theta(r)\Delta x(r)}{\Delta\theta(r)}.$$

The foregoing equation yields

$$\frac{\Delta\varphi(r)}{\varphi(r)} = \frac{\Delta\theta(r)}{\theta(r)}$$

or

$$\boldsymbol{W}(r)\{\alpha_1\alpha_2\} = 0.$$

Since neither $\boldsymbol{W}(i)$ nor $\{\alpha_1\alpha_2\}$ can vanish, our assumption that $\phi_n(x)$ and $\theta_n(x)$ vanish simultaneously leads to a contradiction.

If neither $\phi_n(x)$ nor $\theta_n(x)$ vanish twice on X, our theorem is trivial. Suppose that c and d are two zeros of $\phi_n(x)$ and assume that $\theta_n(x)$ does not vanish for any x between c and d. The function $\phi_n(x)/\theta_n(x)$ is continuous

on $c \leqq x \leqq d$ and has a derivative everywhere on this interval except at the finite subset of E_n that lies on $c \leqq x \leqq d$.

$$(d/dx)[\phi_n(x)/\theta_n(x)] = [\boldsymbol{\theta}(i)\Delta\varphi(i) - \varphi(i)\Delta\boldsymbol{\theta}(i)]/\theta_n^2(x)\Delta x(i)$$
$$= \boldsymbol{W}(i)\{\alpha_1\alpha_2\}/\theta_n^2(x)\Delta x(i) \quad \text{on } x_{in} < x < x_{i+1,n}.$$

Since $\boldsymbol{W}(i)$ and $\{\alpha_1\alpha_2\}$ both maintain the same sign on X, we see that $\phi_n(x)/\theta_n(x)$ is a monotonic, continuous, non-identically zero, function of x on $c \leqq x \leqq d$ and therefore cannot vanish at both c and d. This contradicts our hypothesis that $\phi_n(x)$ vanishes at both of these points and hence $\theta_n(x)$ must vanish at least once between every pair of zeros of $\phi_n(x)$. If we apply the above argument to $\theta_n(x)/\phi_n(x)$ we arrive at the conclusion that between every pair of zeros of $\theta_n(x)$ there is at least one zero of $\phi_n(x)$. Hence the zeros of $\theta_n(x)$ and $\phi_n(x)$ separate each other on X.

COROLLARY 1: *If* $|\boldsymbol{B}(i)| > 0$ *and* $\boldsymbol{B}_{12}(i)$ *maintains the same sign on* E_n *and is different from zero on* E_n, *then the zeros of* $u_{1n}(x)$ and $v_{1n}(x)$ *separate each other on* X.*

COROLLARY 2: *If* $|\boldsymbol{B}(i)| > 0$ *and* $\boldsymbol{B}_{21}(i)$ *maintains the same sign and is different from zero on* E_n, *then the zeros of* $u_{2n}(x)$ *and* $v_{2n}(x)$ *separate each other on* X.

THEOREM 2.3: *If* $\mathbf{y}(i)$ *is a non-identically vanishing solution of* (2.1) *and* $\varphi(i) = \boldsymbol{\alpha}(i) \cdot \mathbf{y}(i)$, $\boldsymbol{\theta}(i) = \boldsymbol{\beta}(i) \cdot \mathbf{y}(i)$, *where* $\alpha_1^2(i) + \alpha_2^2(i) \neq 0$, $\beta_1^2(i) + \beta_2^2(i) \neq 0$, $\{\alpha_1\alpha_2\} \neq 0$, $\{\beta_1\beta_2\} \neq 0$, *then the zeros of* $\phi_n(x)$ *and* $\theta_n(x)$ *separate each other on* X *provided that* $\boldsymbol{R}(i) < 0$ *and* $\boldsymbol{S}(i)$ *maintains the same sign on* E_n, *where*

$$\boldsymbol{R}(i) = \{\boldsymbol{B}_{22}(i)[\alpha_2(i+1)\beta_1(i) - \alpha_1(i)\beta_2(i+1)] + \boldsymbol{B}_{12}(i)[\alpha_1(i+1)\beta_1(i)$$
$$- \alpha_1(i)\beta_1(i+1)] + \boldsymbol{B}_{21}(i)[\alpha_2(i)\beta_2(i+1)$$
$$- \alpha_2(i+1)\beta_2(i)]$$
$$+ \boldsymbol{B}_{11}(i)[\alpha_2(i)\beta_1(i+1) - \alpha_1(i+1)\beta_2(i)]\}^2$$
$$- 4\boldsymbol{d}(i)\boldsymbol{d}(i+1)\,|\boldsymbol{B}(i)|$$
$$\boldsymbol{S}(i) = \boldsymbol{B}_{11}(i)[\beta_1(i)\alpha_1(i+1) - \beta_1(i+1)\alpha_1(i)] + [\alpha_2(i+1)\beta_1(i)$$
$$- \beta_2(i+1)\alpha_1(i)]\boldsymbol{B}_{21}(i)$$
$$\boldsymbol{d}(i) = \alpha_2(i)\beta_1(i) - \alpha_1(i)\beta_2(i).$$

* If (2.1) is specialized to the forms used by these writers, this corollary becomes the theorem on the separation of the nodes that was first proved by Porter, *loc. cit.*, p. 63, and later by Moulton, *Annals of Mathematics*, Vol. 13 (1911), p. 137; Carmichael, *American Journal of Mathematics*, Vol. 43 (1921), p. 80, and Dunkel, *Bulletin of the American Mathematical Society*, Vol. 32, p. 333.

Proof: The same proof that was given under theorem 2.2 shows that $\phi_n(x)$ and $\theta_n(x)$ have only a finite number of zeros on X. To show that $\phi_n(x)$ and $\theta_n(x)$ cannot both vanish at the same point of X, we assume that they both vanish at $x = t$, $x_{rn} < t \leqq x_{r+1,n}$. Solving the equations $\phi_n(t) = 0$, $\theta_n(t) = 0$, we get

$$t = x_{rn} - \frac{\varphi(r)\Delta x(r)}{\Delta\varphi(r)} = x_{rn} - \frac{\theta(r)\Delta x(r)}{\Delta\theta(r)}$$

or $\boldsymbol{Q}(r) = 0$, where $\boldsymbol{Q}(i)$ is a quadratic form in $\boldsymbol{y}_1(i)$ and $\boldsymbol{y}_2(i)$ that has its discriminant equal to $\boldsymbol{R}(i)$ and the coefficient of $\boldsymbol{y}_1^2(i)$ equal to $\boldsymbol{S}(i)$. The conditions on $\boldsymbol{R}(i)$ and $\boldsymbol{S}(i)$ are sufficient to make this quadratic form definite on X and hence the equation $\boldsymbol{Q}(r) = 0$ cannot be satisfied for any real values of $\boldsymbol{y}_1(r)$ and $\boldsymbol{y}_2(r)$. This contradiction proves that $\phi_n(x)$ and $\theta_n(x)$ cannot both vanish at the same point of X.

If neither of the two functions vanish twice then our theorem is meaningless. Let c and d be two zeros of $\phi_n(x)$ and assume that $\theta_n(x)$ does not vanish between $x = c$ and $x = d$. The function $\phi_n(x)/\theta_n(x)$ will be continuous on $c \leqq x \leqq d$ and will have a derivative everywhere on this interval except at a finite number of points.

$$\frac{d}{dx}\left[\frac{\phi_n(x)}{\theta_n(x)}\right] = \frac{\boldsymbol{Q}(i)}{\theta_n^2(x)\Delta x(i)} \text{ on } x_{in} < x < x_{i+1,n}.$$

Since $\boldsymbol{Q}(i)$ has the same sign on X, we see that $\phi_n(x)/\theta_n(x)$ is a continuous, monotonic, non-identically zero, function on $c \leqq x \leqq d$ and hence cannot vanish at both c and d. Since this contradicts our hypothesis that c and d are zeros of $\phi_n(x)$ it follows that $\theta_n(x)$ must vanish at least once between c and d. A similar consideration of $\theta_n(x)/\phi_n(x)$ proves that between every pair of zeros of $\theta_n(x)$ there is at least one zero of $\phi_n(x)$ and hence it follows that the zeros of $\theta_n(x)$ and $\phi_n(x)$ separate each other on X.

COROLLARY: *The following special cases of theorem 2.3 are important:**

(a). *If* $\beta_2(i) \equiv 0$ *and* $\beta_1(i) \equiv 1$, *we get separation of the zeros of* $y_{1n}(x)$ *and* $\phi_n(x)$.

(b). $\beta_2(i) \equiv 1$ *and* $\beta_1(i) \equiv 0$ *yields separation of the zeros of* $y_{2n}(x)$ *and* $\phi_n(x)$.

(c). $\alpha_1(i) \equiv 1$, $\alpha_2(i) \equiv 0$, $\beta_1(i) \equiv 0$, $\beta_2(i) \equiv 1$ *yields separation of the zeros of* $y_{1n}(x)$ *and* $y_{2n}(x)$.

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* The conditions simplify very much in these special cases. Simpler conditions are derived if we assume that $\alpha_1 \neq 0$, $\beta_1 \neq 0$ but in doing this we lose some of the above special cases.

THEOREM 2.4: *If* $\mathbf{y}(i)$ *is a non-identically vanishing solution of* (2.1) *and* $\varphi(i) = \alpha(i) \cdot \mathbf{y}(i)$, *where* $\alpha_2(i) > 0$ *and* $\{\alpha_1\alpha_2\} \neq 0$, *then the product* $y_{in}(x)\phi_n(x)$ *cannot vanish more than once on* X *provided* $\mathbf{y}(i)$ *and* $\varphi(i \pm 1)$ *do not both vanish for any* i, $T(x,i) < 0$ *on* XE_n* *and* $P(x,i)$ *maintains the same sign on* XE_n, *where*

$$P(x,i) = [1 + 2\boldsymbol{A}_{11}(i)\,(x - x_{in})]\,[(\Delta\boldsymbol{\beta}(i)/\Delta x(i)) + \boldsymbol{\beta}(i+1)\boldsymbol{A}_{11}(i) + \boldsymbol{A}_{21}(i)] + \boldsymbol{\beta}(i)\boldsymbol{A}_{11}(i)$$

$$T(x,i) = \{\boldsymbol{A}_{12}(i)[\boldsymbol{\beta}(i+1) + \boldsymbol{\beta}(i) + 2(x - x_{in})[2\boldsymbol{A}_{11}(i)\boldsymbol{\beta}(i+1) + (\Delta\boldsymbol{\beta}(i)/\Delta x(i)) + \boldsymbol{A}_{21}(i)]\} + \boldsymbol{A}_{22}(i) + \boldsymbol{A}_{11}(i) + 2(x - x_{in})\boldsymbol{A}_{11}(i)\boldsymbol{A}_{22}(i)\}^2 - 4P(x,i)\boldsymbol{A}_{12}(i)[1 + 2(x - x_{in})\{\boldsymbol{\beta}(i+1)\boldsymbol{A}_{12}(i) + \boldsymbol{A}_{22}(i)\}] \quad \text{and} \quad \boldsymbol{\beta}(i) = \alpha_1(i)/\alpha_2(i).$$

Proof: Let $\theta(i) = \varphi(i)/\alpha_2(i)$. Clearly the product $y_{1n}(x)\theta_n(x)$ will have the same number of zeros on X as the product $y_{1n}(x)\phi_n(x)$. The derivatives of these products exist everywhere on X except for a finite number of points. The derivative of $y_{1n}\theta_n$ can be written as a quadratic form in $\mathbf{y}_1(i)$ and $\mathbf{y}_2(i)$ and this quadratic form has $T(x,i)$ for its discriminant and $P(x,i)$ as a coefficient of one of its squared terms. The conditions on $T(x,i)$ and $P(x,i)$ are sufficient to insure that this quadratic form have the same sign on X. It follows that the product $y_{1n}(x)\theta_n(x)$ is continuous and monotonic on X and since it cannot vanish for two consecutive values of i, it cannot vanish more than once on X.†

COROLLARY 1: *If we let* $\alpha_1(i) \equiv 0$, *we get conditions for the non-oscillation of* $y_{1n}(x)$ *and* $y_{2n}(x)$.

COROLLARY 2: *If* $\boldsymbol{A}_{11}(i) \equiv \boldsymbol{A}_{22}(i) \equiv 0$, $\boldsymbol{A}_{12}(i) \equiv 1/\boldsymbol{K}(i)$, $\boldsymbol{A}_{21}(i) \equiv \boldsymbol{G}(i)$, *the conditions for the non-oscillation of the solution become* $\boldsymbol{G}(i)$ *of constant sign on* E_n *and* $0 < \boldsymbol{G}(i)/\boldsymbol{K}(i) < 4/[\Delta x(i)]^2$. *These conditions serve to insure non-oscillation for the self-adjoint system when the points of* E_n *are sufficiently close together.*

3. *Existence Theorems.* In this section we will consider the second order system

$$(3.1) \quad \cdot \qquad \Delta\mathbf{y}(i)/\Delta x(i) = \boldsymbol{z}(i)/\boldsymbol{K}(i,\lambda)$$
$$\Delta\boldsymbol{z}(i)/\Delta x(i) = \boldsymbol{G}(i,\lambda)\mathbf{y}(i) + \boldsymbol{G}(i,\lambda)\,\Delta x(i)\boldsymbol{z}(i)/\boldsymbol{K}(i,\lambda)$$

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* This means for every x on X and for every x_{in} on E_n.

† If the requirement that $y(i)$ and $\varphi(i \pm 1)$ not both vanish is removed, the above argument shows that the product $y_{1n}(x)\phi_n(x)$ cannot vanish for two distinct values of x without vanishing for every x that lies between these values.

where $\boldsymbol{K}$ and $\boldsymbol{G}$ are continuous functions of λ on L: $L_1 < \lambda < L_2$ for each i. System (3.1) can be written in the form

$$\frac{\Delta}{\Delta x(i)}\left[\boldsymbol{K}(i,\lambda)\frac{\Delta \boldsymbol{y}(i)}{\Delta x(i)}\right] - \boldsymbol{G}(i,\lambda)\boldsymbol{y}(i+1) = 0. \tag{3.2}$$

Equations (3.1) show immediately that $\boldsymbol{y}(i,\lambda)$ and $\boldsymbol{z}(i,\lambda)$ are continuous functions of λ on L. The continuity of the zeros of the functions $y_n(x,\lambda)$, $z_n(x,\lambda)$, and $\phi_n(x,\lambda)$ follows from the continuity of $\boldsymbol{y}(i,\lambda)$, $\boldsymbol{z}(i,\lambda)$ and $\boldsymbol{\varphi}(i,\lambda)$ in λ. We note that if $f_n(t,\lambda) = 0$, then

$$t(\lambda) = x_{in} - \frac{\boldsymbol{f}(i,\lambda)\Delta x(i)}{\Delta \boldsymbol{f}(i,\lambda)} \qquad \text{when } x_{in} < t < x_{i+1,n}.$$

THEOREM 3.1: *If $\boldsymbol{y}(i,\lambda)$ is a solution of (3.2) and $\boldsymbol{K}(i,\lambda) > 0$, then the zeros of $y_n(x,\lambda)$ move to the left on X as λ increases provided that $\boldsymbol{G}(i,\lambda)/\boldsymbol{K}(i,\lambda)$ and $\boldsymbol{y}(1,\lambda)/\boldsymbol{y}(0,\lambda)$ are non-increasing functions of λ and these ratios are not both constant over any part of the λ-interval.*

Proof: This theorem was proved by Fort (*loc. cit.*, pp. 8-9). He stated his hypotheses in a slightly different form but they are equivalent to the above.

THEOREM 3.2: *Any equation of the type*

$$\boldsymbol{N}(i,\lambda)(\Delta^2\boldsymbol{y}(i)/\Delta x(i)^2) + \boldsymbol{L}(i,\lambda)(\Delta \boldsymbol{y}(i)/\Delta x(i)) + \boldsymbol{M}(i,\lambda)\boldsymbol{y}(i) = 0$$

*where $\boldsymbol{N}$ and $[1 + \boldsymbol{M}/\boldsymbol{N}[\Delta x(i)]^2 - \boldsymbol{L}/\boldsymbol{N}\Delta x(i)]$ are different from zero, can be written in the form of (3.2).**

Proof: Let

$$\boldsymbol{K}(i,\lambda) = \prod_0^{i-1}\left[1 + \frac{\boldsymbol{M}(j,\lambda)}{\boldsymbol{N}(j,\lambda)}\,\Delta x(j)^2 - \frac{\boldsymbol{L}(j,\lambda)}{\boldsymbol{N}(j,\lambda)}\Delta x(j)\right]^{-1}$$

$$\boldsymbol{G}(i,\lambda) = -\boldsymbol{M}(i,\lambda)/\boldsymbol{N}(i,\lambda)\boldsymbol{K}(i+1,\lambda)$$

and our equation takes the desired form.

THEOREM 3.3: *Under the hypotheses of theorem 3.1, $\boldsymbol{y}(i,\lambda)$ changes signs when it vanishes.*

Proof: Let $\boldsymbol{y}(i,\boldsymbol{\lambda}) = 0$. Neither $\boldsymbol{y}(i-1,\boldsymbol{\lambda})$ nor $\boldsymbol{y}(i+1,\boldsymbol{\lambda})$ can vanish and since these functions are continuous functions of λ, we can determine an interval, $\lambda' < \boldsymbol{\lambda} < \lambda''$, such that neither $\boldsymbol{y}(i-1,\lambda)$ nor $\boldsymbol{y}(i+1,\lambda)$ can vanish on this interval. Equation (3.2) gives

$$\boldsymbol{K}(i-1,\boldsymbol{\lambda})\boldsymbol{y}(i-1,\boldsymbol{\lambda}) + \boldsymbol{K}(i,\boldsymbol{\lambda})\boldsymbol{y}(i+1,\boldsymbol{\lambda}) = 0$$

* Porter (*loc. cit.*, p. 60) proved this theorem by means of a change of variable.

and since $\boldsymbol{K}(i,\lambda) > 0$ on E_nL, $\boldsymbol{y}(i-1,\lambda)$ and $\boldsymbol{y}(i+1,\lambda)$ differ in signs. Since these two functions do not vanish on the λ-interval that we have chosen, $\boldsymbol{y}(i+1,\lambda)$ and $\boldsymbol{y}(i-1,\lambda)$ differ in signs for every λ on $\lambda' < \lambda < \lambda''$. There is only one zero of $y_n(x,\lambda)$ on $x_{i-1,n} \leqq x \leqq x_{i+1,n}$ and this zero moves to the left as λ increases from λ' to λ''. For $\lambda' < \lambda < \boldsymbol{\lambda}$, $\boldsymbol{y}(i,\lambda)$ has the same sign as $\boldsymbol{y}(i+1,\lambda)$ and for $\boldsymbol{\lambda} < \lambda < \lambda''$, $\boldsymbol{y}(i,\lambda)$ has the sign of $\boldsymbol{y}(i-1,\lambda)$ and hence $\boldsymbol{y}(i,\lambda)$ changes sign when it vanishes.

COROLLARY: *$y_n(x,\lambda)$ can neither gain nor lose zeros except at the ends of the x-interval.*

Let $\boldsymbol{\varphi}(i,\lambda) = \boldsymbol{\alpha}(i,\lambda)\boldsymbol{z}(i,\lambda) - \boldsymbol{\beta}(i,\lambda)\boldsymbol{y}(i,\lambda)$, where $\boldsymbol{\alpha}(i,\lambda) \neq 0$ and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are continuous functions of λ on L for each fixed i. Compute $\Delta\boldsymbol{\varphi}(i,\lambda)$ and eliminate $\boldsymbol{y}$ and $\boldsymbol{z}$ between this equation and the above equation for $\boldsymbol{\varphi}$. An application of the transformation given in theorem 3. 2 yields

$$\frac{\Delta}{\Delta x(i)}\left[\boldsymbol{H}(i,\lambda)\,\frac{\boldsymbol{\varphi}(i,\lambda)}{\Delta x(i)}\right] - \boldsymbol{J}(i,\lambda)\boldsymbol{\varphi}(i+1,\lambda) = 0 \tag{3.3}$$

where

$$\boldsymbol{H}(i,\lambda) = \prod_{0}^{i=1}\frac{\{\boldsymbol{\theta}(j)\}}{\{\boldsymbol{\theta}(j+1)\}}, \quad \boldsymbol{\theta}(j) = \boldsymbol{\beta}(j)/\boldsymbol{\alpha}(j)$$

$$\{\boldsymbol{\theta}(j)\} = \boldsymbol{\theta}(j)[\boldsymbol{G}(j)\Delta x(j) - \boldsymbol{\theta}(j+1)]/\boldsymbol{K}(j) + \boldsymbol{G}(j) - \frac{\Delta\boldsymbol{\theta}(j)}{\Delta x(j)}$$

$$\boldsymbol{J}(i,\lambda) = -\boldsymbol{H}(i+1,\lambda)\left[\frac{\boldsymbol{\theta}(i+2) - \boldsymbol{G}(i+2)\Delta x(i+2)}{\boldsymbol{K}(i+1)} - \frac{\boldsymbol{\theta}(i)\{\boldsymbol{\theta}(i+1)\}}{\boldsymbol{K}(i)\{\boldsymbol{\theta}(i)\}}\right]$$

The above transformation demands that $\{\boldsymbol{\theta}(i)\}$ be different from zero. We can make the zeros of $\phi_n(x,\lambda)$ move to the left as λ increases by imposing the same conditions on $\boldsymbol{H}$ and $\boldsymbol{J}$ that were imposed on $\boldsymbol{K}$ and $\boldsymbol{G}$ by theorem 3. 1. We will suppose that such conditions are satisfied and that the hypotheses of theorem 2. 3 are met by $\boldsymbol{\varphi}$ and $\boldsymbol{\theta}$ for each fixed λ on L, where $\boldsymbol{\theta}(i,\lambda) = \boldsymbol{\nu}(i,\lambda)\boldsymbol{z}(i,\lambda) - \boldsymbol{\delta}(i,\lambda)\boldsymbol{y}(i,\lambda)$ and $\boldsymbol{\nu}$ and $\boldsymbol{\delta}$ are continuous functions of λ on L.

Definition: By a characteristic number of a difference or differential system we mean a value of the parameter for which the system has a non-identically vanishing solution.

THEOREM 3. 4: *The equation* (3. 1) *together with the boundary conditions*

$$\boldsymbol{\varphi}(0,\lambda) = \boldsymbol{\varphi}(n,\lambda) = 0 \tag{3.4}$$

has $n-1$ *characteristic numbers,* $p_1, p_2, \cdots p_{n-1}$, such that if $\mathbf{y}(i, p_j)$, $\mathbf{z}(i, p_j)$ *is the solution that goes with* p_j *then* $y_n(x, p_j)$ *has exactly* j *zeros on* X, *provided condition* (3.5) *is satisfied.*

$$(3.5)\qquad \frac{\min \boldsymbol{G}(i)}{min\, \boldsymbol{K}(i)} \geqq 0 \quad \text{for each fixed } \lambda \text{ on } L_1 < \lambda < \mu_1, \text{ and}$$

$$\frac{\max \boldsymbol{G}(i)}{max\, \boldsymbol{K}(i)} \leqq \frac{-4}{[\max \Delta x(i)]^2} \text{ for every fixed } \lambda \text{ on } \mu_2 < \lambda < L_2,$$

where μ_1 and μ_2 are fixed points such that $L_1 < \mu_1 < \mu_2 < L_2$.

Proof: We can choose a solution $\mathbf{y}(i, \lambda)$, $\mathbf{z}(i, \lambda)$ of (3.1) such that $\mathbf{y}(0, \lambda) = \boldsymbol{\alpha}(0, \lambda)$ and $\mathbf{z}(0, \lambda) = \boldsymbol{\beta}(0, \lambda)$ for all values of λ on L. This solution satisfies the condition $\boldsymbol{\varphi}(0, \lambda) = 0$ for all values of λ on L. Condition (3.5) insures that $\varphi_n(x, \lambda)$ does not vanish on $a < x \leqq b$ for any λ on $L_1 < \lambda < \mu_1$ while this same condition, together with the fact that the zeros of $\phi_n(x, \lambda)$ and $y_n(x, \lambda)$ separate each other on X, assures us that $\phi_n(x, \lambda)$ has $n-1$ zeros on X for every fixed λ on $\mu_2 < \lambda < L_2$.* If we extend the interval X indefinitely and let the points of this extended interval be at a distance of one unit from each other, and if we let $\boldsymbol{K}(i, \lambda) \equiv 1$ and $\boldsymbol{G}(i, \lambda) \equiv -4$ outside of X, we see that $y_n(x)$ and $\phi_n(x)$ will have as many zeros as you please on this extended interval. As λ increases the zeros of $\phi_n(x, \lambda)$, Which are continuous functions of λ, move into the X interval at $x = b$ and when $\phi_n(b, \lambda) = 0$ we get the characteristic numbers of the system (3.1), (3.4). Since no zeros of $y_n(x, \lambda)$ and $\phi_n(x, \lambda)$ are lost and since the only point where zeros enter the interval X is at $x = b$, our oscillation theorem follows.

COROLLARY: $\phi_n(x, p_j)$ *has exactly* j *zeros on* X.

THEOREM 3.5: *If conditions* (3.5) *are met, there exists a set of characteristic numbers,* $q_1, q_2, \cdots, q_n - 1$, *for the general self-adjoint system* (3.1), (3.6), *where*

$$(3.6)\qquad \begin{aligned} \boldsymbol{\varphi}(0, \lambda) &= \boldsymbol{\varphi}(n, \lambda) \\ \boldsymbol{\theta}(0, \lambda) &= \boldsymbol{\theta}(n, \lambda) \end{aligned}$$

and

$$\boldsymbol{\alpha}(0, \lambda)\boldsymbol{\delta}(0, \lambda) - \boldsymbol{\beta}(0, \lambda)\boldsymbol{\gamma}(0, \lambda) = \boldsymbol{\alpha}(n, \lambda)\boldsymbol{\delta}(n, \lambda) - \boldsymbol{\beta}(n, \lambda)\boldsymbol{\gamma}(n, \lambda).$$

Case I. $\boldsymbol{\alpha}(0)\boldsymbol{\delta}(0) - \boldsymbol{\beta}(0)\boldsymbol{\gamma}(0) = 0.$

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* Compare (3.1) with the equations: $\mathbf{y}(i+2) + 2\mathbf{y}(i+1) + \mathbf{y}(i) = 0$ and $\mathbf{y}(i+2) - 2\mathbf{y}(i+1) + \mathbf{y}(i) = 0$.

This case can be reduced back to the case considered under theorem 3.4 by making linear combinations of the boundary conditions.

Case II. $\alpha(0)\delta(0) - \beta(0)\gamma(0) \neq 0$. *In this case we make* $\alpha(0)\delta(0) - \beta(0)\gamma(0) = 1$ *by division.*

It is easy to see that there is always a solution y_1, z_1 for the system (3.1), (3.7) and there is always a solution y_2, z_2 for (3.1), (3.8), where

$$(3.7)\qquad \varphi_1(0,\lambda) = 0 \qquad\qquad (3.8)\qquad \varphi_2(0,\lambda) = 1$$
$$\theta_1(0,\lambda) = 1 \qquad\qquad \theta_2(0,\lambda) = 0$$

and

$$\varphi_1 = \alpha z_1 - \beta y_1,\ \varphi_2 = \alpha z_2 - \beta y_2,\ \theta_1 = \gamma z_1 - \delta y_1,\ \theta_2 = \gamma z_2 - \delta y_2.$$

If we solve (3.7) and (3.8) we get

$$z_1(0) = \beta(0),\ y_1(0) = \alpha(0),\ z_2(0) = -\delta(0),\ y_2(0) = -\gamma(0).$$

y_1, z_1 and y_2, z_2 form a fundamental system of solutions of (3.1) since $W(i, \lambda) = 1$. We also have

$$(3.9)\quad \varphi_1(n)\theta_2(n) - \varphi_2(n)\theta_1(n) = -W(n)[\alpha(n)\delta(n) - \beta(n)\gamma(n)] = -1.$$

We write the solution of (3.1) as

$$y(i, \lambda) = c_1 y_1(i, \lambda) + c_2 y_2(i, \lambda)$$
$$z(i, \lambda) = c_1 z_1(i, \lambda) + c_2 z_2(i, \lambda)$$

and substitute this general solution into the boundary conditions.

$$c_1[\varphi_1(0) - \varphi_1(n)] + c_2[\varphi_2(0) - \varphi_2(n)] = 0$$
$$c_1[\theta_1(0) - \theta_1(n)] + c_2[\theta_2(0) - \theta_2(n)] = 0.$$

In order that there exist a solution for the system (3.1), (3.6) it is necessary that $D(\lambda)$ be zero, where $D(\lambda)$ is the determinant of coefficients of c_1 and c_2 in the above equations. If we make use of (3.7), (3.8), and (3.9), $D(\lambda)$ takes the form

$$D(\lambda) = -2 + \varphi_2(n, \lambda) + \theta_1(n, \lambda).$$

If all of the elements of $D(\lambda)$ vanish for a value of λ, then that value of λ is a double characteristic number of system (3.1), (3.6), since in that case both of the solutions y_1, z_1 and y_2, z_2 satisfy the system. Let us consider the case where $D(\lambda)$ is of rank $\geqq 1$. Let $p_1, p_2, \cdots p_{n-1}$, be the

characteristic numbers of the system (3.1), (3.4) and substitute p_j into (3.9). This gives

$$\varphi_2(n, p_j) = 1/\theta_1(n, p_j)$$

and

$$D(p_j) = [1 - \theta_1(n, p_j)]^2/\theta_1(n, p_j)$$

so that $D(p_j)$ either has the same sign as $\theta_1(n, p_j)$ or it is zero in which case p_j is a characteristic number of our system.

By theorem 4.3 we have that the zeros of $\varphi_{1n}(x, \lambda)$ and $\theta_{1n}(x, \lambda)$ separate each other on X for a fixed λ. Also, $\varphi_1(n, \lambda)$ and $\theta_1(n, \lambda)$ change signs when they vanish. Since $\varphi_1(n, p_j) = 0$, it follows that $\theta_1(n, p_j) \neq 0$ and that it is either positive for $j = 1, 3, \cdots$, and negative for $j = 2, 4, \cdots$, or negative for $j = 1, 3, 5, \cdots$, and positive for $j = 2, 4, \cdots$. Since $D(p_j)$ has the same sign as $\theta_1(n, p_j)$ or else it is zero, it follows that $D(\lambda)$ has at least one root between (or coincident with one of the two) each pair of characteristic numbers of system (3.1), (3.4). This establishes our existence theorem.

4. *The Passage to the Limit.* In this section we will let the number of points in E_n increase indefinitely in such a way that these points form a *fine* subdivision of the interval X, where by a *fine* subdivision we mean one such that the length of the maximum subdivision approaches zero as the number of these subdivisions becomes infinite. All integrals are to be taken in the sense of Lebesgue.

1°. *The Method of Successive Approximations.* Consider the m^{th} order differential system

$$(dy_j/dx) - \sum_{r=1}^{r=m} A_{jr} y_r = \beta_j \qquad j = (1, m)$$

$$y_j(a) = \alpha_j$$

which we will write in the form*

$$(dy/dx) - A \cdot y = \beta \tag{4.1}$$

$$y(a) = \alpha$$

following the notation described in the introduction of this paper. A_{jk} and β_j are assumed to be Lebesgue summable functions of x on X and α_j is a constant. We set up the difference system

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* An alternate form of (4.1) is $y(x) = \int_a^x [A(t) \cdot y(t) + \beta(t)]dt + \alpha$.

$$\frac{\Delta \boldsymbol{y}(i)}{\Delta x(i)} - \boldsymbol{A}(i) \cdot \boldsymbol{y}(i) = \boldsymbol{\beta}(i) \qquad \boldsymbol{y}(0) = \alpha \tag{4.2}$$

which takes the form

$$\boldsymbol{y}(i+1) = \sum_{r=0}^{r=i} [\boldsymbol{B}(r) \cdot \boldsymbol{y}(r) + \varphi(r)] + \alpha \qquad \boldsymbol{y}(0) = \alpha \tag{4.3}$$

when we let $\boldsymbol{B}(i) = \boldsymbol{A}(i)\Delta x(i)$ and $\varphi(i) = \boldsymbol{\beta}(i)\Delta x(i)$. Let

$$\boldsymbol{y}_0(i+1) = \alpha + \sum_{r=0}^{r=i} \varphi(r), \qquad \boldsymbol{y}_0(0) = \alpha$$

and let $\boldsymbol{y}_1(i)$ be determined by

$$\boldsymbol{y}_1(i+1) = \alpha + \sum_{r=0}^{r=i} [\boldsymbol{B}(r) \cdot \boldsymbol{y}_0(r) + \varphi(r)], \qquad \boldsymbol{y}_1(0) = \alpha.$$

In general let $\boldsymbol{y}_k(i)$ be determined by

$$\boldsymbol{y}_k(i+1) = \alpha + \sum_{r=0}^{r=i} [\boldsymbol{B}(r) \cdot \boldsymbol{y}_{k-1}(r) + \varphi(r)], \qquad \boldsymbol{y}_k(0) = \alpha.$$

We wish to show that $\lim_{k\to\infty} \boldsymbol{y}_k(i)$ exists and is the unique solution of system (4.2). Let

$$\boldsymbol{z}_0(i) = \boldsymbol{y}_0(i)$$

and

$$\boldsymbol{z}_k(i) = \boldsymbol{y}_k(i) - \boldsymbol{y}_{k-1}(i) \text{ for } k \geqq 1.$$

We see immediately that if $k > 0$

$$\boldsymbol{z}_k(i+1) = \sum_{r=0}^{r=i} \boldsymbol{B}(r) \cdot \boldsymbol{z}_{k-1}(r)$$

while $\boldsymbol{y}_k(i+1)$ is given by

$$\boldsymbol{y}_k(i+1) = \sum_{j=0}^{j=k} \boldsymbol{z}_j(i+1).$$

Our problem is to show that the series

$$\boldsymbol{z}_0(i+1) + \boldsymbol{z}_1(i+1) + \cdots$$

converges to a sum $\boldsymbol{y}(i+1)$ which is the solution of system (4.2). Let $\boldsymbol{u}(r)$ be the sum of the quantities $|\boldsymbol{B}_{pj}(r)|$, α and $\varphi^*(r)$ be the respective sums of the quantities $|\alpha_j|$ and $|\varphi_j(r)|$, and let the constant M be defined by

$$M = m\,[\alpha + \sum_{r=0}^{r=n} \varphi^*(r)].$$

The foregoing series is made up of m component series and when we say that it converges we mean that each of its components converges. We have

$$|\boldsymbol{z}_{0j}(i+1)| \leqq M$$

$$|\boldsymbol{z}_{1j}(i+1)| \leqq (mM/1!) \sum_{r=0}^{r=i} \boldsymbol{u}(r)$$

and in general

$$|\, z_{kj}(i+1)\,| \leqq (m^k M/k!)\,[\sum_{r=0}^{r=i} u(r)]^k \tag{4.4}$$

where z_{kj} is any one of the m components of z_k. In order to prove that (4.4) is valid for all positive integral values of k, we note that it is true for $k=0$ and $k=1$. If there are indices for which it is not true, then there is a first such index and we will call that one k. Now

$$|\, z_{kj}(i+1)\,| = |\sum_{r=0}^{r=i}\sum_{p=1}^{p=m} B_{jp}(r) z_{k-1,\,p}(r)\,| \leqq \sum_{r=0}^{r=i} u(r) \sum_{p=1}^{p=m} |\, z_{k-1,\,p}(r)\,|$$

and making use of the fact that (4.4) is valid for $k-1$ and $z_k(0)=0$ for $k \geqq 1$, we get

$$|\, z_{kj}(i+1)\,| \leqq [m^k M/(k-1)!] \sum_{r=1}^{r=i} u(r)\,[\sum_{s=0}^{s=r-1} u(s)]^{k-1}$$

Let

$$v(r-1) = \sum_{s=0}^{s=r-1} u(s),$$

then $\Delta v(r-1) = u(r)$ and

$$\sum_{r=1}^{r=i} u(r)\,[\sum_{s=0}^{s=r-1} u(s)]^{k-1} = \sum_{r=1}^{r=i} [v(r-1)]^{k-1}\,\Delta v(r-1).$$

Since

$$\begin{aligned}\Delta[v(r-1)]^k &= [v(r-1) + \Delta v(r-1)]^k - [v(r-1)]^k \\ &= k[v(r-1)]^{k-1} v(r-1) + k(k-1)/2!\,[v(r-1)]^{k-2}\,[\Delta v(r-1)]^2 \\ &\qquad + \cdots + [\Delta v(r-1)]^k\end{aligned}$$

and both $v(r-1)$ and $\Delta v(r-1)$ are positive or zero, we have

$$[v(r-1)]^{k-1}\Delta v(r-1) \leqq \Delta\,[v(r-1)]^k/k$$

and hence

$$\begin{aligned}|\, z_{kj}(i+1)\,| &\leqq (m^k M/k!) \sum_{r=1}^{r=i} \Delta[v(r-1)]^k \\ &\leqq (m^k M/k!)\,[v(i)]^k - [v(0)]^k \\ &\leqq (m^k M/k!)\,[\sum_{r=0}^{r=i} u(r)]^k.\end{aligned}$$

Therefore (4.4) is valid for k which contradicts our hypothesis that k is an index for which it is not valid and hence (4.4) is valid for all positive integral values of k.

Let C_k be defined by

$$C_k = (m^k M/k!)\,[\sum_{r=0}^{r=n-1} u(r)]^k.$$

Since $\boldsymbol{u}(r) \geqq 0$, it follows from (4.4) that $|\boldsymbol{z}_{kj}(i+1)| \leqq C_k$ independent of the value of i. Since the series of constants C_k is absolutely convergent, it follows that each of the component series of the series

$$\sum_{k=0}^{k=\infty} \boldsymbol{z}_k(i+1)$$

is uniformly convergent. Let $\boldsymbol{y}(i+1)$ denote the vector whose components are the respective sums of these series.

THEOREM 4.1: *$\boldsymbol{y}(i)$ is the unique solution of system* (4.2).

Proof: The condition $\boldsymbol{y}(0) = \alpha$ is satisfied since $\boldsymbol{z}_0(0) = \alpha$, $\boldsymbol{z}_k(0) = 0$ for $k \geqq 1$. Also

$$\begin{aligned}\boldsymbol{y}(i+1) - \boldsymbol{y}(i) &= \lim_{k\to\infty} [\boldsymbol{y}_k(i+1) - \boldsymbol{y}_k(i)] \\ &= \lim_{k\to\infty} [\boldsymbol{B}(i) \cdot \boldsymbol{y}_{k-1}(i) + \boldsymbol{\varphi}(i)] \\ &= \boldsymbol{B}(i) \cdot \boldsymbol{y}(i) + \boldsymbol{\varphi}(i)\end{aligned}$$

and dividing by $\Delta x(i)$ we get the equation of (4.2). The uniqueness of this solution follows from the fact that the difference between $\boldsymbol{y}(i)$ and any other solution of (4.2) would satisfy the system that is obtained from (4.2) by replacing α and $\boldsymbol{\beta}(i)$ with zeros. However, this system has no solution other than the identically vanishing one as is easily seen when we calculate $\boldsymbol{y}(1)$, $\boldsymbol{y}(2), \cdots, \boldsymbol{y}(n)$ successively by means of the equations (4.3).

Let us now define the coefficients of system (4.2) as follows:

$$\boldsymbol{A}_{rj}(i) = \frac{\int_{x_{in}}^{x_{i+1,n}} A_{rj}(t)\, dt}{\Delta x(i)}, \qquad \boldsymbol{\beta}_j(i) = \frac{\int_{x_{in}}^{x_{i+1,n}} \beta_j(t)\, dt}{\Delta x(i)}$$

where $A_{rj}(t)$ and $\beta_j(t)$ are coefficients of the differential system (4.1). Let n become infinite in such a way that the subdivision of the interval X is *fine.*

Definition: By a solution of the differential equation (4.1) we mean a set of absolutely continuous functions that satisfy this equation almost everywhere on X.

THEOREM 4.2: *If x is a fixed point of X and if the index p varies in such a way that at every stage of the limiting process $x_{pn} \leqq x \leqq x_{p+1,n}$, then $\lim_{n\to\infty} \boldsymbol{y}(p) = y(x)$, where $y(x)$ is the unique solution of system* (4.1).

Lemma: If $z_{0j}(x) = \alpha_j + \int_a^x \beta_j(t)\,dt$

$$z_{kj}(x) = \int_a^x \sum_{s=1}^{s=m} A_{js}(t) z_{k-1,s}(t)\,dt, \text{ for } k = 1, 2, \cdots,$$

then $\lim_{n\to\infty} z_{kj}(p+1) = z_{kj}(x)$ *for every* x *on* X *and for* $k = 0, 1, 2, \cdots$.

Proof of Lemma: Since $z_{kj}(p+1)$ consists of only m terms, where m is finite and independent of n, it is sufficient to show that the typical term of $z_{kj}(p+1)$ has the corresponding term of $z_{kj}(x)$ for its limit. We wish to show that

$$\lim_{n\to\infty} [\alpha_j + \sum_{r=0}^{r=p} \varphi_j(r)] = \alpha_j + \int_a^x \beta_j(t)\,dt$$

(4.5)

$$\lim_{n\to\infty} [\sum_{r=0}^{r=p} B_{js}(r)\, z_{k-1,s}(r) = \int_a^x A_{js}(t) z_{k-1,s}(t)\,dt, \text{ for } k \geqq 1$$

(4.5) is valid for all values of k and for every x on X. It is valid for $k = 0$, since

$$\sum_{r=0}^{r=p} \varphi_j(r) = \int_a^{x_{p+1,n}} \beta_j(t)\,dt = \int_a^x \beta_j(t)\,dt + \int_x^{x_{p+1,n}} \beta_j(t)\,dt$$

and the limit of the last integral is zero. If there are values of k for which (4.5) is not valid for every x on X, then let r be the first such value and let p be associated with a value of x for which (4.5) is not valid. Consider the difference

$$\sum_{i=0}^{i=p} B_{js}(i) z_{r-1,s}(i) - \int_a^x A_{js}(t)\, z_{r-1,s}(t)\,dt \tag{4.6}$$

which we will write in the form $I_1 + I_2 + I_3$, where

$$I_1 = \int_a^x A_{js}(t)\,[z^{(p)}{}_{r-1,s}(t) - z_{r-1,s}(t)]\,dt$$

$$I_2 = \sum_{i=0}^{i=p} B_{js}(i) z_{r-1,s}(i) - \int_a^{x_{p+1,n}} A_{js}(t) z^{(p)}{}_{r-1,s}(t)\,dt$$

$$I_3 = \int_x^{x_{p+1,n}} A_{js}(t)\, z^{(p)}{}_{r-1,s}(t)\,dt$$

and

$$z^{(p)}{}_{r-1,s}(t) = z_{r-1,s}(i) \text{ on } x_{in} < t < x_{i+1,n}.$$

Let $f_n(t) = A_{js}(t)\,[z^{(p)}{}_{r-1,s}(t) - z_{r-1,s}(t)]$. It follows from the fact that (4.5) is valid for $k = r-1$ that $\lim_{n\to\infty} f_n(t) = 0$ for every t on X. Further-

more, of $g(x)$ and $\beta^*(x)$ denote the respective sums of $|A_{jk}(x)|$ and $|\beta_j(x)|$, then these quantities are summable and it is easy to see that

$$u(i) = \int_{x_{in}}^{x_{i+1,n}} g(t)\,dt \text{ and } M = m\,[\alpha + \int_a^b \beta^*(t)\,dt.$$

We have then

$$|z_{kj}(i+1)| \leqq (m^k M/k!)\,[\int_a^b g(t)dt]^k = C_k$$

independent of n and i. We also note that $z_{kj}(x)$ is continuous on X and hence it is bounded. Let G be the sum of this bound and C_k. The sequence of functions $f_n(t)$ is less than the summable function $g(t)G$ for all n and for every t on X. We may now apply Lebesgue's theorem * concerning the limit of the integral of a sequence of functions to get $\lim_{n\to\infty} I_1 = 0$.

Now consider I_2. We see immediately that

$$I_2 = \sum_{i=0}^{i=p} \int_{x_{in}}^{x_{i+1,n}} A_{js}(t)\,[z_{r-1,s}(i) - z^{(p)}{}_{r-1,s}(t)]\,dt \equiv 0$$

for all values of n since $z^{(p)}{}_{r-1,s}(t) = z_{r-1,s}(i)$ on $x_{in} < t < x_{i+1,n}$.
Also

$$I_3 = z_{r-1,s}(p) \int_x^{x_{p+1,n}} A_{js}(t)\,dt \leqq G \int_x^{x_{p+1,n}} g(t)\,dt$$

and since $\int_x^{x_{p+1,n}} g(t)\,dt$ is a continuous function of $x_{p+1,n}$ and $\lim_{n\to\infty} x_{p+1,n} = x$, we have $\lim_{n\to\infty} I_3 = 0$.

We have shown that the limit of the expression (4. 6) is zero and hence that (4. 5) is valid for $k = r$ and for every x on X. This contradicts our hypothesis that (4. 5) was not valid for $k = r$ and some x on X and hence it follows that (4. 5) is valid for all values of x on X and for $k = 0, 1, 2, \cdots$.

Proof of Theorem 4. 2: Theorem 4. 2 follows from an application of the above lemma and a theorem due to Tannery.† We have

$$z_{0j}(p) + z_{1j}(p) + \cdots + z_{kj}(p)$$

such that $|z_{kj}(p)| \leqq C_k$ independent of n and the series of constants C_k is convergent. Also, from the above lemma we have that $\lim_{n\to\infty} z_{kj}(p) = z_{kj}(x)$. Hence by Tannery's theorem

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* *Leçons sur l'intégration etc.*, Paris, 1904, p. 114.

† *Introduction à l'étude des fonctions*, Paris, 1866, Sec. 183.

$$\lim_{n\to\infty} y_j(p) = \lim_{n\to\infty} \sum_{k=0}^{k=\infty} z_{kj}(p) = \lim_{n\to\infty} \sum_{k=0}^{k=\infty} z_{kj}(x) = y_j(x).$$

By taking the limit of system (4.3) we get

$$y(x) = \int_a^x [A(t) \cdot y(t) + \beta(t)]\, dt + \alpha$$

for every x on X, or

$$y'(x) = A(x) \cdot y(x) + \beta(x)$$
$$y(a) = \alpha$$

almost everywhere on X. $y(x)$ is a solution of (4.1). The uniqueness of this solution can be established directly by a method that is the same as the one used to establish this property when the coefficients are continuous.

2°. *The Coefficients Contain a Parameter.* Consider system (4.1) where the coefficients are continuous functions of λ on L for each fixed x on X, summable functions of x on X for each fixed λ on L, and bounded numerically for all x and λ on XL by a function $M(x)$ that is summable on X. Let α_j be a continuous function of λ on L. Let the coefficients of the difference system (4.2) be continuous in λ on L for each x_{1n} of E_n, and bounded numerically for all λ and x_{in} on LE_n by a function $u(i)$ such that $\lim_{n\to\infty} u(i)$ exists almost everywhere and is summable on X.

We obtain the solution of the difference system by exactly the same method that was used in the preceding case. Each of the functions $z_{kj}(i, \lambda)$ is continuous in λ on L and since all of the series converge uniformly with respect to λ and i, it follows that the limit functions $y_j(i, \lambda)$ are continuous in λ on L.

If the coefficients of the difference system are defined as they were in the preceding case, we see that all of the above hypotheses are met. These coefficients are continuous * in λ on L and bounded numerically by the function

$$\frac{\int_{x_{in}}^{x_{i+1,n}} M(t)\, dt}{\Delta x(i)}.$$

This function has the summable function $M(x)$ as its limit function almost everywhere on X. All of the proofs of the preceding case apply here and we get

$$\lim_{n\to\infty} y(p, \lambda) = y(x, \lambda)$$

* See Carathéodory, *Vorlesungen über Reele Funktionen*, Teubner, Berlin, 1918, Chapter XI.

where $y(x, \lambda)$ is continuous in the two variables x and λ and is the unique solution of the differential system.

3°. *A Non-Linear System.* Consider the system

$$dy/dx = f(x, y) \qquad y(a) = \alpha \tag{4.7}$$

where $f(x, y)$ is the m-dimensional vector whose jth component is

$$f_j[x, y_1(x), y_2(x), \cdots, y_m(x)].$$

$f(x, y)$ satisfies the conditions given by Carathéodory * in his general existence theorem. The corresponding difference system is

$$\frac{\Delta y(i)}{\Delta x(i)} = \boldsymbol{f}[i, \boldsymbol{y}(i)] \qquad y(0) = \alpha. \tag{4.8}$$

If we define

$$\boldsymbol{f}_j[i, \boldsymbol{y}(i)] = \frac{\int_{x_{in}}^{x_{i+1,n}} f_j[t, \boldsymbol{y}(i)]\, dt}{\Delta x(i)}$$

and apply the same method of proof that was used in treating the linear system, we get that the limit of the system (4.8) and its solution is the system (4.7) and its solution respectively.

4°. *Application of Theorem* 4.2. Theorem 4.2 enables us to carry over the results of parts 2 and 3 to the limiting differential systems. It follows from the absolute continuity of indefinite integrals that the coefficients of the difference systems as defined in this section, have the summable coefficients of the differential systems as their limit functions almost everywhere on X. Our conditions on $\boldsymbol{B}(i)$ go over into similar conditions on $B(x)$. Theorem 2.1 goes over into a corresponding theorem for the differential system when we note that theorem 4.1 carries $\boldsymbol{W}(i)$ over into the wronskian of the differential system. If we define $\phi(x)$ to be equal to $\alpha(x) \cdot y(x)$, where α_1 and α_2 are absolutely continuous functions on X, and if we let $\boldsymbol{\alpha}_1(i)$ and $\boldsymbol{\alpha}_2(i)$ respectively be the values of $\alpha_1(x)$ and $\alpha_2(x)$ at x_{in}, then

$$\lim_{n\to\infty} \frac{\{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2\}}{\Delta x(i)} = \{\alpha_1(x)\alpha_2(x)\} = \alpha_2^2(x)A_{21}(x) + \alpha_1(x)\alpha_2(x)[A_{11}(x) - A_{22}(x)] + \alpha_2(x)\alpha_1'(x) - \alpha_1(x)\alpha_2'(x) - \alpha_1^2(x)A_{12}(x)$$

* *Loc. cit.*, p. 672 ff.

almost everywhere on X. If we specialize our differential system to the self-adjoint form by letting $A_{11}(x) = A_{22}(x) = 0$, $A_{21}(x) = G(x)$, $A_{12}(x) = 1/K(x)$, we see immediately that $\{\alpha_1(x)\alpha_2(x)\}$ is Bôcher's * $\{\phi_1\phi_2\}$ relation. Theorem 2. 2 goes over into a corresponding theorem for the limiting system when we require that $\{\alpha_1(x)\alpha_2(x)\}$ be different from zero. We can replace the proof for a finite number of zeros by one that does not depend on n. Theorems 2. 3 and 2. 4 go over for the differential system without introducing any difficulties. $\boldsymbol{R}(i)/[\Delta x(i)]^2$ and $\boldsymbol{S}(i)/[\Delta x(i)]$ go over into similar expressions. We note that for the self-adjoint system, the conditions that the zeros of $y_1(x)$ and $y_2(x)$ separate each other become that $K(x)$ be positive and $G(x)$ be negative. The condition that the solution of the self-adjoint system be non-oscillatory is that K and G both be positive.

5. *The Green's Function and the Adjoint System.* In this section we deal with mth order systems. We write these systems in vector and tensor form for the sake of compactness.

Consider the differential system

$$(5.1) \qquad \begin{aligned} (dy/dx) - A(x) \cdot y &= \beta(x) \\ \phi[y(a)] - \phi[y(b)] &= 0 \end{aligned}$$

where $\phi[y(x)] = R(x) \cdot y(x)$, $R(x)$ is a tensor of absolutely continuous functions, A and β are summable functions of x on X. Let (5. 1′) denote the homogeneous system obtained from (5. 1) by making $\beta(x) \equiv 0$. Consider the corresponding difference system

$$(5.2) \qquad \begin{aligned} \frac{\Delta \boldsymbol{y}(i)}{\Delta x(i)} - \boldsymbol{A}(i) \cdot \boldsymbol{y}(i) &= \boldsymbol{\beta}(i) \\ \boldsymbol{\varphi}[\boldsymbol{y}(0)] - \boldsymbol{\varphi}[\boldsymbol{y}(n)] &= 0 \end{aligned}$$

where $\boldsymbol{\varphi}[\boldsymbol{y}(i)] = \boldsymbol{R}(i) \cdot \boldsymbol{y}(i)$. Let (5. 2′) be the system obtained from (5. 2) by making $\boldsymbol{\beta}(i) \equiv 0$.

If the system (5. 2′) has no solution other than the identically vanishing one, it is said to be incompatible. We suppose that (5. 2′) is incompatible and proceed to define its Green's tensor † in the following manner: Let $\boldsymbol{Y}(i)$, be a tensor of solutions of (5. 2′) such that the (mth ordered) determinant of $\boldsymbol{Y}(i)$, $|\boldsymbol{Y}(i)|$, is different from zero for all values of i. Let x_{sn} be a point of E_n. We seek to determine a tensor $\boldsymbol{G}(i, s)$ in such a way that its vector components $\boldsymbol{G}_j(i, s)$ will have the following properties:

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* *Leçons sur les Méthodes de Sturm*, Paris, 1917, p. 45.

† For a discussion of the Green's tensor for differential systems, see my paper "The Green's Functions for Systems of Differential Equations," *Annals of Mathematics*, Vol. 28 (1927), No. 3, pp. 291-300.

1. $\boldsymbol{G}_j(i,s)$ as a function of i satisfies the boundary conditions of (5. 2′).

2. $\boldsymbol{G}_j(i,s)$ satisfies the equation of (5. 2′) for every i except for $i=s-1$ and it satisfies this equation when $\boldsymbol{G}_{jj}(s,s)$ is decreased by unity.*

Let

$$\begin{aligned} \boldsymbol{G}(i,s) &= \boldsymbol{Y}(i)\cdot C(s) \qquad \text{for } i<s \\ \boldsymbol{G}(i,s) &= \boldsymbol{Y}(i)\cdot D(s) \qquad \text{for } i\geqq s \end{aligned}$$

and try to determine tensors C and D in such a way that $\boldsymbol{G}$ has the above properties. In order to have property 2, it must satisfy

$$\boldsymbol{Y}(s)\cdot[D(s)-C(s)]=E,$$

where E is the unit tensor and is defined so that $E_{ij}=0$ for $i\neq j$, $E_{jj}=1$. Since $|\boldsymbol{Y}(i)|\neq 0$, we can solve this equation for the unique value of $D-C$. This gives

$$D(s)-C(s)=\boldsymbol{Y}^{-1}(s)\cdot E=\boldsymbol{Y}^{-1}(s)$$

where $\boldsymbol{Y}^{-1}$ is the inverse tensor of $\boldsymbol{Y}$ and is defined by $\boldsymbol{Y}\cdot\boldsymbol{Y}^{-1}=E$. Substitute $\boldsymbol{G}_j$ into the boundary conditions and let Q be the tensor such that $Q_{kj}=\varphi_k[\boldsymbol{Y}_j(0)]-\varphi_k[\boldsymbol{Y}_j(n)]$. Let S be the tensor whose general element is $S_{kj}=\varphi_k[\boldsymbol{Y}_j(n)]$.

$$Q\cdot C=S\cdot\boldsymbol{Y}^{-1}(s)$$

and since system (5. 2′) is incompatible, we may solve for C and substitute its value into the above equation for $D-C$ to get

$$\begin{aligned} C(s) &= Q^{-1}\cdot S\cdot\boldsymbol{Y}^{-1}(s) \\ D(s) &= Q^{-1}\cdot S\cdot\boldsymbol{Y}^{-1}(s)+\boldsymbol{Y}^{-1}(s). \end{aligned} \tag{5.3}$$

Since D and C are uniquely determined by (5. 3) it follows that the Green's tensor is uniquely determined by the properties 1 and 2.

THEOREM 5. 1: *The unique solution of system* (5. 2) *is* $\boldsymbol{y}(i)$, *where*

$$\boldsymbol{y}_j(i)=\sum_{s=1}^{s=n}\boldsymbol{G}_j(i,s)\cdot\boldsymbol{\beta}(s-1)\Delta x(s-1).$$

Proof: It is evident that $\boldsymbol{y}(i)$ satisfies the boundary conditions since $\boldsymbol{G}_j(i,s)$ satisfies these conditions. Substitute $\boldsymbol{y}(i)$ into the jth equation of (5. 2)

$$\sum_{s=1}^{s=n}\frac{\Delta\boldsymbol{G}_j(i,s)}{\Delta x(i)}-\sum_{r=1}^{r=m}\boldsymbol{A}_{jr}(i)\boldsymbol{G}_r(i,s)]\cdot\boldsymbol{\beta}(s-1)\Delta x(s-1)=\boldsymbol{\beta}_j(i).$$

Using property 2 of $\boldsymbol{G}_j$ we see that the left hand side reduces to $\boldsymbol{\beta}_j(i)$ and hence the equation is satisfied. The uniqueness of this solution follows when

* This is equivalent to requiring that $G(s,s)-G(s,s)=E$.

we note that the difference between any two solutions of (5.2) would satisfy (5.2′) which is assumed incompatible.

THEOREM 5.2: *If the coefficients of system* (5.2) *are defined as they were in part* 4 *and if the number of points in* E_n *is allowed to increase indefinitely in such a way that the subdivision of* X *is fine, then the limit of* $\boldsymbol{G}(i,s)$ *is* $G(x,t)$, *where* $G(x,t)$ *is the Green's tensor for the system* (5.1′), *provided the systems* (5.2′) *are incompatible for all values of* n.

Proof: It follows from theorem 4.2 that $\boldsymbol{Y}(i)$ goes over in the limit to $Y(x)$, where $Y(x)$ is a tensor of solution of the equation (5.1′) and $|\,Y(x) \neq 0$ on X. Similarly $\boldsymbol{Y}^{-1}(i)$ goes over into $Y^{-1}(x)$. Since S and Q are made up of fixed finite linear combinations of the values of the elements of $\boldsymbol{Y}(i)$ at a and b, it follows that Q, Q^{-1} and S go over into corresponding tensors for the differential system. An examination of (5.3) shows that $C(s)$ and $D(s)$ go over into $C(t)$ and $D(t)$ respectively, and hence

$$\lim_{n\to\infty} \boldsymbol{G}(i,s) = G(x,t).$$

Let

$$L[\boldsymbol{y}(i)] = \frac{\Delta \boldsymbol{y}(i)}{\Delta x(i)} \boldsymbol{A}(i) \cdot \boldsymbol{y}(i)$$

and determine a function $\boldsymbol{v}(i)$ in such a way that

$$\boldsymbol{v}(i) \cdot L[\boldsymbol{y}(i)] = \frac{\Delta[\boldsymbol{y}(i) \cdot \boldsymbol{v}(i-1)]}{\Delta x(i)}.$$

We see immediately that $\boldsymbol{v}(i)$ must satisfy the equation *

$$M[\boldsymbol{v}(i)] = \frac{\Delta \boldsymbol{v}(i+1)}{\Delta x(i)} + \boldsymbol{v}(i) \cdot \boldsymbol{A}(i) = 0. \tag{5.4}$$

Equation (5.4) is defined to be the adjoint of the equation $L[\boldsymbol{y}(i)] = 0$.

Lagrange's identity is

$$\boldsymbol{v}(i) \cdot L[\boldsymbol{y}(i)] + \boldsymbol{y}(i) \cdot M[\boldsymbol{v}(i)] = \frac{\Delta \boldsymbol{y}(i) \cdot \boldsymbol{v}(i-1)]}{\Delta x(i)}$$

while Green's formula is

$$\sum_{i=1}^{i=n-1} \{\boldsymbol{v}(i) \cdot L[\boldsymbol{y}(i)] + \boldsymbol{y}(i) \cdot M[\boldsymbol{v}(i)]\} \Delta x(i) = \boldsymbol{y}(i) \cdot \boldsymbol{v}(i-1) \Big|_{i=1}^{i=n-1}.$$

A repetition of the argument used in proving theorem 4.2 carries the adjoint system (5.4) into the corresponding limiting differential system and with it the Lagrange identity and Green's formula are carried over.

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* L and M are m-dimensional vectors.

On the Motion of a Rigid Body about a Fixed Point in Space of Constant Curvature.

By James Pierpont.

1. This problem, in a more general form, has been treated by Clifford and Heath * for elliptic space, and in two papers by de Francesco.† The first two papers are so general that it is difficult to have much of an idea what actually takes place. Clifford uses his biquaternions, which must first be learned to understand his paper. Heath's treatment rests on six homogeneous line co-ordinates tied by two relations. De Francesco employs sometimes four and sometimes six co-ordinates. The problem is, however, carried out in much greater detail.

In the present paper we restrict the problem to motion about a fixed point; moreover to get formulae we can handle we have supposed the body moves in a constant field of force. This latter assumption is the one made in classical mechanics. The lines of force due to gravity are of course neither parallel nor constant, yet the dimensions of the gyroscope or top are so small relative to the earth that no sensible error is made. Under these assumptions the problem admits a solution of remarkable simplicity, if we make use of a few ideas I have developed in earlier papers.‡

2. We assume first a Euclidean space, whose points are defined by rectangular co-ordinates x, y, z for which the metric is given by

$$d\sigma^2 = dx^2 + dy^2 + dz^2.$$

Let us set

$$r^2 = x^2 + y^2 + z^2, \quad \lambda = 4R^2 + r^2, \quad \mu = 4R^2 - r^2.$$

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* W. K. Clifford, "Motion of a Solid in Elliptic Space," *London Mathematical Society*, 1874. First published from posthumous notes in his *Mathematical Papers*, p. 374 (1882); R. S. Heath, "On the Dynamics of a Rigid Body in Elliptic Space," *Philosophical Transactions of the Royal Society of London*, Vol. 175 (1885), p. 281.

†D. de Francesco, "Alcuni Problemi di Meccanica in uno Spazio a tre Dimensioni di Curvatura constante," *Atti della Reale Academia di Scienze di Napoli* (2), Vol. 10 (1900), No. 4 and 9.

‡ "The Geometry of Riemann and Einstein," *American Mathematical Monthly*, Vol. 30 (1923), p. 26; "Non-Euclidean Geometry from Non-Projective Standpoint," *Transactions of the Sixth International Congress* (August, 1924), Vol. 1, p. 117.

Using the same co-ordinates, we may define an elliptic space by

$$ds = \frac{4R^2 d\sigma}{\lambda} \tag{1}$$

and a hyperbolic one by

$$ds = \frac{4R^2 d\sigma}{\mu},$$

where $R > 0$ is called the space constant.

We shall restrict ourselves to elliptic space for the sake of clearness; it is not difficult to make the necessary modifications for hyperbolic space. For brevity let us read *e-* and *E-* as Euclidean and elliptic, respectively.

The points of *E*-space lie within the *e*-sphere $\mu = 0$. Two diametral *e*-points on $\mu = 0$ are regarded as one *E*-point. The length of a segment of an *E*-curve is given by $\int ds$ and the curve is an *E*-straight, if the first variation of this integral is zero. *E*-straights appear to an *e*-observer as circles cutting $\mu = 0$ in diametral points. *E*-straights are closed curves each having the length πR. Two points A, B, on an *E*-straight determine two segments. The length of the shorter one is the length of the segment $A.B$. We write $\delta = \text{dist}\,(A, B)$. *E*-straights thru the origin 0 are also *e*-straights. An *E*-plane is an *e*-sphere cutting $\mu = 0$ along a great circle. The angle between two *E*-straights, planes, etc., is the same in *E*-measure as in *e*-measure, *i. e.*, an *e*-observer and an *E*-observer measuring an angle would get the same result.

If we introduce four new variables

$$z_1 = 4R^2x/\lambda, \quad z_2 = 4R^2y/\lambda, \quad z_3 = 4R^2z/\lambda, \quad z_4 = R\mu/\lambda, \tag{2}$$

satisfying the identical relation $z_1^2 + z_2^2 + z_3^2 + z_4^2 = R^2$, we find that any linear relation $u_1z_1 + u_2z_2 + u_3z_3 + u_4z_4 = 0$ defines an *E*-plane. In these variables $ds^2 = dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2$. The four planes $z_1 = 0, \cdots, z_4 = 0$ define a tetrahedron τ. We find $z_i = R \sin(\delta_i/R)$, where δ_i is the length of the perpendicular from z on the face $z_1 = 0$, $i = 1, 2, 3, 4$. The distance δ between two points a, b or the *E*-length of the segment $a \cdot b$ is given by

$$R^2 \cos(\delta/R) = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4.$$

If $\delta = \pi(R/2)$ we say the points a and b are associated, $a \sim b$.

In the z co-ordinates the equations of an *E*-straight are

$$z_i = a_i \cos(s/R) + b_i \sin(s/R), \qquad (i = 1, \cdots, 4).$$

Here a, b are two points on the straight and $a \sim b$ and s is dist (a, z), measured in a given sense.

Let us subject the points z of E-space to a linear transformation.

$$z'_i = a_{i1}z_1 + \cdots + a_{i4}z_4, \qquad (i = 1, \cdots, 4). \tag{3}$$

whose determinant $a = |a_{ij}| = +1$. If the scalar product of two rows are orthogonal i. e. if

$$\begin{aligned} a_{i1}a_{j1} + \cdots + a_{i4}a_{j4} &= 0, \quad \text{if } i \neq j \\ &= 1, \qquad i = j, \end{aligned} \tag{4}$$

we find the distance between two points is the same before and after the transformation. Such a transformation is called an E-displacement. Let

$$\begin{aligned} x' &= \alpha_1 x + \beta_1 y + \gamma_1 z \\ y' &= \alpha_2 x + \beta_2 y + \gamma_2 z \\ z' &= \alpha_3 x + \beta_3 y + \gamma_3 z \end{aligned} \tag{5}$$

define an e-rotation about 0. If we multiply these equations on the left by $4R^2/\lambda'$

$$\lambda' = 4R^2 + x'^2 + y'^2 + z'^2$$

and on the right by $4R^2/\lambda$ observing that $\lambda = \lambda'$ we have

$$\begin{aligned} z'_1 &= \alpha_1 z_1 + \beta_1 z_2 + \gamma_1 z_3 + 0 z_4 \\ z'_2 &= \alpha_2 z_1 + \beta_2 z_2 + \gamma_2 z_3 + 0 z_4 \\ z' &= \alpha_3 z_1 + \beta_3 z_2 + \gamma_3 z_3 + 0 z_4 \\ z'_4 &= 0 z_1 + 0 z_2 + 0 z_3 + z_4. \end{aligned} \tag{6}$$

Thus to the e-rotation corresponds a linear transformation in E-space. Its determinant is $+1$, its coefficients satisfy the orthogonal relation (4).

It defines an E-displacement. All this is more easily seen by noting that (6) leaves $d\sigma$ and λ unaltered and hence ds. Since e-straights and planes thru 0 are E-straights and planes, and since angles have the same value in both measures we have a complete picture of an E-rotation about 0. This does not hold for any other point.

Let us apply this to small e-rotations about 0. We see these as compounded precisely as in e-geometry.

Thus three small rotations ω_x, ω_y, ω_z about the x, y, z axes produce the displacement.

$$\begin{aligned} \delta z_1 &= -z_2\omega_z + z_3\omega_y \\ \delta z_2 &= \quad z_1\omega_z - z_3\omega_x \\ \delta z_3 &= -z_1\omega_y + z_2\omega_x \\ \delta z_4 &= 0. \end{aligned} \tag{7}$$

Consider the point a whose co-ordinates are

$$a_1 = \frac{4R^2\omega_x}{\lambda}, \quad a_2 = \frac{4R^2\omega_y}{\lambda}, \quad a_3 = \frac{4R^2\omega_z}{\lambda}, \quad a_4 = (R\mu/\lambda),$$

where

$$\omega^2 = \omega_x{}^2 + \omega_y{}^2 + \omega_z{}^2, \quad \lambda = 4R^2 + \omega^2, \quad \mu = 4R^2 - \omega^2.$$

If l is the E-straight thru 0 and a, the e-displacement (7) is an E-rotation about l thru the angle ω.

Let $\xi\eta\zeta$ be an orthogonal e-set of axes thru 0 fixed in the body and defined by the customary scheme (8)

(8)

	x	y	z
ξ	α_1	β_1	γ_1
η	α_2	β_2	γ_2
ζ	α_3	β_3	γ_3

having the determinant $+1$. The $\xi\eta\zeta$ axes define a new tetrahedron τ'. If we denote the E-co-ordinates of a point referred to τ and τ' by $(z_1, \cdots z_4)$ and $(\zeta_1, \cdots \zeta_4)$ respectively, we have

$$\begin{aligned} z_1 &= \alpha_1\zeta_1 + \alpha_2\zeta_2 + \alpha_3\zeta_3 \\ z_2 &= \beta_1\zeta_1 + \beta_2\zeta_2 + \beta_3\zeta_3 \\ z_3 &= \gamma_1\zeta_1 + \gamma_2\zeta_2 + \gamma_3\zeta_3 \\ z_4 &= \zeta_4. \end{aligned}$$

Let us give the body the displacement (7). Then

$$\delta z_1 = \zeta_1\delta\alpha_1 + \zeta_2\delta\alpha_2 + \zeta_3\delta\alpha_3, \quad \text{etc.} \tag{9}$$

By *purely analytical transformations* we find as in e-space

$$\delta\alpha_1 = \gamma_1\omega_y - \beta_1\omega_z, \quad \delta\alpha_2 = \gamma_2\omega_y - \beta_2\omega_z, \quad \delta\alpha_3 = \gamma_3\omega_y - \beta_3\omega_z, \text{ etc.} \tag{10}$$

If we set

$$\begin{aligned} p = \omega_\xi &= \alpha_1\omega_x + \beta_1\omega_y + \gamma_1\omega_z \\ q = \omega_\eta &= \alpha_2\omega_x + \beta_2\omega_y + \gamma_2\omega_z \\ r = \omega_\zeta &= \alpha_3\omega_x + \beta_3\omega_y + \gamma_3\omega_z \end{aligned} \tag{11}$$

we find that ω_ξ, ω_η, ω_ζ are small E-rotations about the ξ, η, ζ axes which together produce the displacement (7).

3. The E-velocity v of a point is defined by

$$v^2 = (ds^2/dt^2) = \dot{z}_1{}^2 + \dot{z}_2{}^2 + \dot{z}_3{}^2 + \dot{z}_4{}^2.$$

The kinetic energy T of a body we define by

$$2T = \sum v^2 dm, \qquad dm = \text{mass of an element.}$$

Now by (9)

$$\dot{z}_1 = \dot{\alpha}_1\zeta_1 + \dot{\alpha}_2\zeta_2 + \dot{\alpha}_3\zeta_3, \quad \text{etc.} \qquad \dot{z}_4 = 0.$$

Hence squaring and using (10) (11) we get,

$$2T = \sum \{p^2(\zeta_2^2 + \zeta_3^2) + q^2(\zeta_1^2 + \zeta_3^2) + r^2(\zeta_1^2 + \zeta_2^2) - 2pq\zeta_1\zeta_2 - 2pr\zeta_1\zeta_3 - 2qr\zeta_2\zeta_3\}\, dm.$$

Set $A = \sum (\zeta_2^2 + \zeta_3^2)dm$, $B = \sum (\zeta_1^2 + \zeta_3^2)dm$, $C = \sum (\zeta_1^2 + \zeta_2^2)dm$,
$D_{12} = \sum \zeta_1\zeta_2 dm$, $D_{13} = \sum \zeta_1\zeta_3 dm$, $D_{23} = \sum \zeta_2\zeta_3 dm$.

Then

$$2T = Ap^2 + Bq^2 + Cr^2 - 2D_{12}pq - 2D_{13}pr - 2D_{23}qr. \tag{12}$$

By a proper choice of the $\xi\eta\zeta$ axes we may make $D_{ij} = 0$ and for such axes

$$2T = Ap^2 + Bq^2 + Cr^2. \tag{13}$$

4. With Killing * we will suppose that the components of a force acting at a point $(z_1, \cdots, z_4)$ are

$$F_i = F \cos \alpha_i, \qquad (i = , \cdots, 4),$$

where α_i is the angle made by a plane thru z and perpendicular to F, with the co-ordinate planes $z_i = 0$. We can give F_i a form which is at times more convenient. For if y is the point on the line F associated with x, we find $F_i = y_i(F/R)$,

and hence $$F^2 = F_1^2 + F_2^2 + F_3^2 + F_4^2.$$

The work done by a small displacement of x to $x' = x + \delta x$ Killing defines by

$$dW = \sum F_i \delta x_i, \qquad (i = 1, \cdots, 4).$$

Hence $$dW = (F/R)\sum y_i dx_i = (F/R)\sum y_i(x_i + dx_i), \text{ as } \sum x_i y_i = 0,$$
$$= (F/R)\sum x_i y_i' = R^2 \cos (r/R), \qquad r = \text{dist } (y, x').$$

Now $$\cos (r/R) = \cos (\pi R/2) \cos (ds/R) + \sin (\pi R/2) \sin (ds/R) \cdot \cos\theta$$
where $$ds = \text{dist } (x, x'), \qquad \theta = \text{Ang } (F, ds).$$

Thus

$$dW = F \cos\theta \cdot ds, \tag{14}$$

as in e-mechanics.

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* W. Killing, "Die Mechanik in den Nicht-Euklidischen Raumformen," *Crelle's Journal*, Vol. 98 (1885), p. 1.

We now find the work done by the force F for a small rotation about 0. Let the line of the force F acting on an element of mass dm at $x = (x_1 \cdots x_4)$ cut $z_4 = 0$ at the point $a = (a_1, \cdots a_4)$.

Any other point x' on this straight has co-ordinates $x'_i = a_i l + x_i m$, i=1 $\cdots$ 4, where

$$l = \frac{\sin [(x'x)/R]}{\sin [(ax)/R]}, \qquad m = \frac{\sin [(ax')/R]}{\sin [(ax)/R]}.$$

Suppose $(ax) = \text{dist } (a, x) = s$, $(xx') = -ds$, then $(ax') = s - ds$. Then

$$l = -\frac{ds/R}{\sin (s/R)}, \qquad m = 1 - \frac{\cos (s/R)}{\sin (s/R)} \cdot (ds/R)$$

neglecting small quantities of second order. Thus

$$dx_k = -[a_k + x_k \cos (s/R)] \frac{ds}{R \sin (s/R)}, \qquad k = 1, 2, 3$$

$$dx_4 = -x_4 \frac{\cos (s/R)}{\sin (s/R)} \cdot (ds/R).$$

These are the changes in the co-ordinates along the line of force F. On the other hand, a small rotation ω_x about the x axis causes x to describe an arc δs; thereby x passes to $x + \delta x$ where

$$\delta x_1 = 0, \qquad \delta x_2 = -x_3 \omega_x, \qquad \delta x_3 = x_2 \omega_x, \qquad \delta x_4 = 0.$$

The angle θ between the arcs ds and δs is given by

$$\cos \theta = \sum (dx_i/ds) \cdot (\delta x_i/\delta s)$$

$$= \frac{a_2 x_3 - a_3 x_2}{R \sin (s/R)} \cdot (\omega_x/\delta s).$$

This in (14) gives the work done by the displacement ω_x.

Let us now suppose (analogous to e-mechanics) that the lines of force F, all issue from a and that F is sensibly proportional to the mass so that $F = mf$. Then for the whole body

$$dW_x = (f\omega_x/R) \{a_2 \sum \frac{x_3 dm}{\sin (s/R)} - a_3 \sum \frac{x_2 dm}{\sin (s/R)}\}$$

with similar expressions for dW_y, dW_z. Now for any rigid body which a physicist in E-space would have to deal with, s is sensibly constant and in the present case equal $\pi R/2$. On this assumption,

$$dW_x = (f\omega_x/R) \{a_2 \sum x_3 dm - a_3 \sum x_2 dm\}$$

and similar expressions of the two other components.

We note next that $|x_i| < R$ hence if $M = \sum dm$

$$\frac{\sum x_i dm}{RM} < 1, \text{ numerically.}$$

We may therefore set this fraction $= \sin(p_i/R)$. If further we set $g_i = R \sin p_i/R$ we have

$$\sum x_i dm = Mg_i, \qquad i = 1, 2, 3; \qquad g_4^2 = 1 - (g_1^2 + g_2^2 + g_3^2).$$

This gives us

$$\begin{aligned} dW_x &= f(M/R)(a_2 g_3 - a_3 g_2)\omega_x, \\ dW_y &= f(M/R)(a_3 g_1 - a_1 g_3)\omega_y, \\ dW_z &= f(M/R)(a_1 g_2 - a_2 g_1)\omega_z. \end{aligned} \tag{15}$$

This shows that the work done by a small rotation about 0 is the same as if the mass M were concentrated at the point g.

As a special case let us take $a = (0, 0, R, 0)$. Then the lines of force F are all perpendicular to the x, y plane which we regard as horizontal and

$$dW = dW_x + dW_y + dW_z = fM(g_2\omega_x - g_1\omega_y). \tag{16}$$

On the other hand ω_x gives g_3 the increment $g_2\omega_x$, ω_y gives g_3 the increment $g_1\omega_y$, while ω_z leaves g unaltered. Thus the rotation $\omega = (\omega_x, \omega_y, \omega_z)$ gives g the increment

$$\delta g_3 = g_1\omega_y + g_2\omega_x.$$

Comparing with (16) we have

$$dW = fM \cdot \delta g_3. \tag{17}$$

5. To get the equations of motion we will use Hamilton's principle *

$$\int (\delta T + \delta W)\, dt = 0. \tag{18}$$

Instead, however, of using supernumerary variables as Killing and Heath do, tied by various relations, we will use independent variables which fix the position of the body. If q is one of these variables (18) gives

$$(d/dt)(\partial T/\partial \dot{q}) - (\partial T/\partial q) = (\partial W/\partial q).$$

If we take Euler's angles as variables we find as in e-mechanics

* W. Killing, *loc. cit.*, p. 13.

$$p = \omega_\xi = \dot\psi \sin\theta \sin\phi + \dot\theta \sin\phi,$$
$$q = \omega_\eta = \dot\psi \sin\theta \cos\phi - \dot\theta \sin\phi,$$
$$r = \omega_\zeta = \dot\psi \cos\theta + \dot\phi.$$

These in (12) or (13) give T in terms of θ, ϕ, ψ. To get dW we use (17) and observe that

$$g_3 = \zeta_1 \sin\phi \sin\theta + \zeta_2 \cos\phi \sin\theta + \zeta_3 \cos\theta$$

where $\zeta_1 \cdots \zeta_4$ are the co-ordinates of g referred to τ'. We see therefore that to an e-observer the body appears to rotate as if its e-center of mass had the e-co-ordinates

$$4R^2 \sum (\xi dm/\lambda), \quad 4R^2 \sum (\eta dm/\lambda), \quad 4R^2 \sum (\zeta dm/\lambda)$$
$$\lambda = 4R^2 + (\xi^2 + \eta^2 + \zeta^2),$$

and the moments of inertia,

$$J = 16R^4 \sum \frac{(\eta^2 + \zeta^2)\, dm}{\lambda^2} \text{ etc}, \qquad \Delta\xi\eta = 16R^4 \sum \frac{\xi\eta dm}{\lambda^2}, \text{ etc.}$$

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On the Prime Ideals of the General Cubic Galois Field.

By Claiborne G. Latimer.

1. In a recent paper, Nowlan * treated the problems of determining the rational integers representable by two cubic forms. Each of these forms is the norm of the general integral algebraic number in a cubic Galois field of class unity. The problem is equivalent to that of determining the rational primes which are factorable in the field. We shall consider this problem for the general cubic Galois field. The discriminants of the fields considered by Nowlan are 7^2 and 9^2 respectively. If we let $D = 7$ or $D = 9$ and let the corresponding field be F, his results, except as to discriminantal divisors, may be stated as follows. *A rational prime p, not a divisor of D, is the product of three conjugate ideals of F if and only if p is a cubic residue of D.*

The discriminant of a cubic Galois field is the square of a rational integer D. If p is a rational prime, not a divisor of $6D$, we shall find a necessary and sufficient condition, in terms of D and p, that p be factorable in the field. When D is 9 or a prime and $p > 3$ our results may be stated exactly as in italics above. It will also be shown that a necessary condition that the field be of class unity is that $D = 9$ or a prime.

It may be shown, in a comparatively simple manner, that certain of Eisenstein's results on the canonical cubic and associated forms may be deduced from the results of this paper.

2. Consider the cubic field F, defined by a root θ_1 of

$$x^3 + ax + b = 0, \tag{1}$$

where a and b are integers.† After an obvious transformation, we may assume that there is no integer $K > 1$ such that a and b are divisible by K^2 and K^3 respectively. If $a = 9a'$, $b = 9b'$, where a' and b' are integers, it may be verified that $\theta' = (6a' + \theta_1^2)/3$ is a root of $x^3 - 3a'^2x - (2a'^3 + 3b'^2) = 0$. The second and third coefficients of this equation are not both divisible by 9 since b' is prime to 3. We may therefore assume that

* *Bulletin of the American Mathematical Society*, 32 (1926), 374-80.

† We shall use "integer" and "prime" to mean "rational integer" and "rational prime" unless some other reference field is mentioned.

(A) *there is no integer* $K > 1$ *such that* a *and* b *are divisible by* K^2 *and* K^3 *respectively; and* a *and* b *are not both divisible by* 9.

By Woronoj's method of determining a basis * of a cubic field, we find that

$$1, \theta_2 = (\theta_1 - \xi)/\delta, \qquad W_1 = (\xi^2 + a + \xi\theta_1 + \theta_1^2)/\sigma\delta^2,$$

constitute a basis, where δ, σ, ξ are as follows. If each of the following conditions is satisfied, $\delta = 3$; otherwise $\delta = 1$.

$$(2) \qquad a \equiv -3 \pmod 9, \qquad \pm b \equiv a + 1 \pmod{27}.$$

σ is the largest integer such that the congruences

$$(3) \qquad x^3 + ax + b \equiv 0 \pmod{\delta^3\sigma^2}, \qquad 3x^2 + a \equiv 0 \pmod{\delta^2\sigma},$$

have a common solution ξ. If d is the discriminant of (1), we find that the discriminant of F is $d/\delta^6\sigma^2$.

If $\delta = 3$, $\theta_1 = 3\theta_2 + \xi$. It will be found that $W_1 = m + (\xi\theta_2 + \theta^2_2)/\sigma$ where by (3_2) m is an integer. If ξ were divisible by 3, $\theta_1/3$ would be integral, contrary to (A). Hence ξ is prime to 3. θ_2 is a root of $x^3 + \xi x^2 + \alpha x + \beta = 0$, where α and β are integers. We have then

THEOREM 1. *The field* F *has a basis in the form*

$$1, \theta, W = (s + t\theta + \theta^2)/\sigma$$

where s *and* t *are rational integers and* θ *is a root of*

$$(4) \qquad f(x) = x^3 + Px^2 + Qx + R = 0,$$

where P, Q, R *are rational integers. If* $P = 0$, *then* $\delta = 1$, $Q = a$, $R = b$, *and if* $P \not\equiv 0$, *then* $P = \xi$, $Q = (3\xi^2 + a)/9$, $R = (\xi^3 + a\xi + b)/27$, *and* P *is prime to* 3.

We shall assume hereafter that the discriminant of $f(x)$ is the square of a rational integer $\Delta > 0$. Then $\Delta = \sigma D$, where D^2 is the discriminant of F. It will be seen later that this assumption is equivalent to assuming that F is a Galois field. We have

$$(5) \qquad -\delta^6\sigma^2D^2 = 4a^3 + 27b^2.$$

It will also be seen later that $3Q - P^2 = fD$, where f is an integer. By Theorem 1, $P = 0$ if and only if fD is divisible by 3, and $3fD = a$ or $9a$ according as $P \not\equiv 0$ or $P = 0$.

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* *Über die ganzen algebraischen Zahlen, die von einer Wurzel der Gleichung dritten Grades abhängen*, St. Petersburg, 1894.

Suppose $\delta = 1$ and that σ is divisible by 3. By (5), $a = 3a'$, a' an integer, and σD is divisible by 9. Then $4a'^3 + b^2 \equiv 0 \pmod{27}$ or $4a'^3 + b^2 \equiv -3 \pmod 9$ according as σD is or is not divisible by 27. The first congruence holds only if the conditions (2) hold, whence $\delta = 3$, contrary to hypothesis. If the second congruence holds, $a = 3a' \equiv -3$, $b \equiv \pm 1 \pmod 9$, and then by (3_1), $\xi^3 - 3\xi \pm 1 \equiv 0 \pmod 9$, which is obviously impossible. We have then (a) of the following lemma. Let k be the g. c. d. of fD and $\sigma = k\sigma'$.

LEMMA 1. (a) *If $\delta = 1$ then σ is prime to 3;*
(b) *$D = 3^j D_1$, where $j = 0$ or 2 and D_1 is prime to 3;*
(c) *k contains no square factor and is prime to $3\sigma'$;*
(d) *k^2 divides a and b;*
(e) *$f = f'k$, $D = D'k$, where f' is prime to σ', and D' is prime to σ.*

If D is divisible by 3, $fD = 3Q = 3a$, $P = 0$, $\delta = 1$ and by (a), σ is prime to 3. Then by (5), D is divisible by 9. If D were divisible by 27, a would be divisible by 9 and hence by (5) the same would be true of b, contrary to (A). This proves (b).

If k were divisible by 3, as before $\delta = 1$ and we would have a contradiction of (a). Hence k is prime to 3. Suppose p^2 divided $\sigma = k\sigma'$, where p is a prime divisor of k. Since 3 $fD = a$ or $9a$, p divides a. By (3_2) a and ξ are divisible by p^2 and p respectively. Then by (3_1) p^3 would divide b, contrary to (A). We have therefore (c). If p is a prime factor of k we find, as in the proof of (c), that a and b are divisible by p and p^2 respectively. Then by (5) p^2 divides a. Since k contains no square factor, (d) follows.

By (a), (d) and (5), k divides $\sigma' D$ and hence by (c) it divides D. Suppose p^2 divided $D = kD'$, where p is a prime factor of k. Then by (5) and $3fD = a$ or $9a$, a and b would be divisible by p^2 and p^3 respectively, contrary to (A). Hence D' is prime to k, and k divides f, since k^2 divides a. f' and D' are prime to σ' by definition of k. This completes the proof of the lemma.

It may be verified that if $\lambda = \alpha_0 + \alpha\theta_1 + \alpha_2\theta_1^2$ then the norm of λ is

$$(6) \qquad \psi(\alpha_i) = \alpha_0^3 + a\alpha_0\alpha_1^2 - b\alpha_1^3 + 3b\alpha_0\alpha_1\alpha_2 - 2a\alpha_0^2\alpha_2 + a^2\alpha_0\alpha_2^2 - ab\alpha_1\alpha_2^2 + b^2\alpha_2^3.$$

Every integer of $F(\theta)$ may be expressed in the forms

$$\lambda = x + y(\theta_1 - \xi)/\delta + z(\xi^2 + a + \xi\theta_1 + \theta_1^2)/\sigma\delta^2 = (\alpha_0 + \alpha_1\theta_1 + \alpha_2\theta_1^2)/\sigma\delta^2$$

where the x, y, z, α_i are integers. Then $\delta^6\sigma^3 N(\lambda) = \psi(\alpha_i)$. Furthermore, by definition of k, (3) and Lemma 1, k divides a and ξ. It follows that α_0 and α_1 are divisible by k. We also note that every integer of $F(\theta)$ may be written in the form $\lambda = (\alpha_0 + \alpha_1\theta + \alpha_2\theta^2)/\sigma$, where the α_i are integers.

3. It is known * that a second root of (4) is

$$\theta^1 = (1/2\sigma D)[2(3Q - P^2)\theta^2 - (9R - 7PQ + 2P^3 + \sigma D)\theta + 4Q^2 - P^2Q - 3PR - PD\sigma]$$

It follows that $F(\theta)$ is a Galois field. We shall now prove

THEOREM 2. *A rational prime p, not a divisor of D, is the product of three distinct conjugate ideals of $F(\theta)$ or is a prime of $F(\theta)$ according as the congruence*

$$f(x) \equiv 0 \pmod{p}, \tag{7}$$

has a solution or has no solution.

It is well-known that every prime ideal in a cubic Galois field is of the first or third degree. Hence every rational prime, not a divisor of D, satisfies one of the conditions of the theorem. Suppose $x = a$ is a solution of (7). If p were a prime of $F(\theta)$ it would divide one of the three factors of $f(a) = (a - \theta)(a - \theta')(a - \theta'')$. This is impossible in view of the basis of $F(\theta)$ given in Theorem 1. The first part of the theorem follows. Suppose (7) has no solution. If $P \neq p$ were a prime ideal factor of p it would have a canonical basis in the form $[p, a - \theta, b - W]$ where a and b are integers. Then the rational integer $f(a) = (a - \theta)(a - \theta')(a - \theta'')$ would belong to P and hence would be divisible by p, contrary to hypothesis. The theorem follows.

By the last sentence in § 2 and the above expression for the integral algebraic number θ' we find

$$3Q - P^2 = fD, \qquad 9R - PQ = hD, \tag{8}$$

where f and h are rational integers. Let p be a prime, not a divisor of $6D$. In (7) set $3x = z - P$. We obtain by means of (8),

$$z^3 + 3fDz + (3h - 2Pf)D = 0 \pmod{p}. \tag{9}$$

The discriminant of (9) is $3^6\sigma^2D^2$. Then from the well-known formula for the discriminant of a reduced cubic we find

$$-f^3D = (\lambda + 3\sigma\rho)(\lambda + 3\sigma\rho^2) \tag{10}$$

* Serret, *Cours d'Algèbre Supérieure*, Vol. 2, 6th ed., p. 467.

where ρ is an imaginary cube root of unity and $2\lambda = 2Pf + 3(\sigma - h)$. By Lemma 1, k divides σ and D and contains no square factor. Hence $\lambda = k\lambda'$, λ' an integer. From (10)

$$-f'^3k^2D' = (\lambda' + 3\sigma'\rho)(\lambda' + 3\sigma'\rho^2).$$

By Lemma 1, σ' is prime to $f'kD'$ and hence it is prime to λ'. Then every prime factor $\neq 3$ of the left side of this equation is the product of two conjugate primes of $F(\rho)$, one and only one of which divides $\lambda' + 3\sigma'\rho$. By Lemma 1, k is prime to 3. Let

$$k = \prod_{i=1}^{m} (s_i + t_i\rho)(s_i + t_i\rho^2),$$

$$D' = 3^j \prod_{i=m+1}^{n} (s_i + t_i\rho)(s_i + t_i\rho^2)$$

where every $s_i + t_i\rho$ ($i = 1, 2, \cdots, n.$) is a primary prime of $F(\rho)$ which divides $\lambda' + 3\sigma'\rho^2$ or $\lambda' + 3\sigma'\rho$ according as $i \leqq m$ or $i > m$. By Lemma 1, $j = 0$ or $j = 2$. Let $-f' = 3^t(A + B\rho)(A + B\rho^2)$ where $A + B\rho$ is the product of the primary prime factors of $F(\rho)$ which divide $\lambda' + 3\sigma'\rho$. Since $3 = (1-\rho)(1-\rho^2) = -\rho^2(1-\rho)^2$, we have

$$\lambda' + 3\sigma'\rho = \pm \rho^e (A + B\rho)^3 (1-\rho)^{j+3t} \times \prod_{i=1}^{m} (s_i + t_i\rho^2)^2 \prod_{i=m+1}^{n} (s_i + t_i\rho), \tag{11}$$

where $e = 0$, 1 or 2. Let $H = D(\lambda + 3\sigma\rho) = k^2D'(\lambda' + 3\sigma'\rho)$. Then

$$H = \rho^{e+2j}(A + B\rho)^3 M^3 \prod_{i=1}^{n} (s_i + t_i\rho)^2 (s_i + t_i\rho^2)$$

where $M = \pm (1-\rho)^{j+t} \prod_{i=1}^{m} (s_i + t_i\rho^2)$. By a result due to Dickson,* (9) and hence (7) has a solution if and only if there exists an integer ζ of $F(\rho)$ such that $\zeta^3 \equiv H \pmod{p}$. By hypothesis p is prime to M and hence we may replace H by $H_1 = H/M^3$ in this criterion.

By (8) and the last sentence of Theorem 1, it is readily shown that a and b are divisible by D_1 of Lemma 1. Then if D_1 were divisible by q^2, q a prime $\neq 3$, σ would be divisible by q from the definition of σ. Therefore by (5), b would be divisible by q^3 contrary to (A). Hence D contains no square factor $\neq 9$. We have then (a) and (b) of

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* *Bulletin of the American Mathematical Society*, Vol. 13 (1907), p. 2.

LEMMA 2. (a) $D=9$ *or* $3^j \prod_{i=1}^{n} (s_i + t_i\rho)(s_i + t_i\rho^2)$, *where* $j=0$ *or* 2 *and the imaginary factors are distinct primary primes of* $F(\rho)$.

(b) *If* p *is a rational prime, not a divisor of* $6D$, (7) *has a solution if and only if there exists an integer* ζ *of* $F(\rho)$ *such that*

$$\zeta^3 \equiv H_1 = \rho^{e+2j}(A+B\rho)^3 \prod_{i=1}^{n} (s_i + t_i\rho)^2 (s_i + t_i\rho^2). \tag{12}$$

(c) $j=0$ *implies* $e=0$; $j=2$ *implies* $e \neq 2$.

To prove (c) write (11) in the form

$$\lambda' + 3\sigma'\rho = \pm \rho^e (1-\rho)^{j+3t}(S+T\rho). \tag{11'}$$

We have $S + T\rho = \pm 1 \pmod 3$ since it is a product of primary primes of $F(\rho)$. Since σ' is prime to 3 and $(1-\rho)^4 = 9\rho^2$, it follows from (11') that $j + 3t \leqq 3$. If $j = t = 0$, the imaginary part of the left side of (11') is divisible by 3 only if $e = 0$. If $j = 0$, $t = 1$, we find from the definition of λ and (11') that

$$-h + \sigma(-3)^{\frac{1}{2}} = \pm 2k\rho^{e+1}[S + T + (2T - S)\rho].$$

By (8_2) h is divisible by 9. But it will be found that the real part of the right side of this equation is divisible by 9 only if $e = 0$. The first part of (c) follows. If $j = 2$ then $t = 0$ and

$$\lambda' + 3\sigma'\rho = \mp 3\rho^{e+1}(S + T\rho).$$

Since T is divisible by 3 and σ' is prime to 3, it follows that $e \neq 2$. This proves the lemma.

4. If π is a prime of $F(\rho)$ such that $N(\pi) = \mu \neq 3$ and if η is an integer of $F(\rho)$ not divisible by π, by Fermat's Theorem

$$\eta^{\mu-1} - 1 = (\eta^{(\mu-1)/3} - 1)(\eta^{(\mu-1)/3} - \rho)(\eta^{(\mu-1)/3} - \rho^2) \equiv 0 \pmod{\pi}.$$

We shall use Eisenstein's symbol, $[\eta/\pi] = 1$, ρ, or ρ^2 according as π divides the first, second or third of the above factors. It is well known that η is a cubic residue of π if and only if the value of the symbol is unity.

From the definition of the symbol we have

$$[\lambda/\pi]^2 = [\lambda'/\pi'], \quad [\lambda/\pi][\eta/\pi] = [\lambda\eta/\pi], \quad [(\eta + \kappa\pi)/\pi] = [\eta/\pi] \tag{13}$$

where the primed letters represent the conjugates of the corresponding

unprimed letters. Furthermore, by the cubic reciprocity law * if η and π are primary primes of $F(\rho)$, then $[\eta/\pi] = [\pi/\eta]$.

If π is a prime of $F(\rho)$, not a divisor of $N(H_1)$, by (13_1) we have $[H'_1/\pi] = [H_1/\pi']^2$. Furthermore, since $N(H_1)$ is a cube, $[H'_1/\pi] = [H_1/\pi]^2$. Hence $[H_1/\pi] = [H_1/\pi']$. Then if $p = \pi\pi'$ or π is a rational prime, not a divisor of $6D$, where π is a primary of $F(\rho)$ not a divisor of $A + B\rho$, a necessary and sufficient condition that (12) have a solution is that $[H_1/\pi] = 1$. We may choose the notation so that π does not divide $A + B\rho$. For if p divided $A + B\rho$, by (11) it would divide σ' and λ', whereas we have seen that they are relatively prime. Therefore by Theorem 2, (b) of Lemma 2 and the cubic reciprocity law we have

THEOREM 3. *If p is a rational prime, not a divisor of $6D$, $p = \pi\pi'$ or $p = \pi$ where π is a primary prime of $F(\rho)$ which does not divide $A + B\rho$. Then a necessary and sufficient condition that p be the product of three conjugate ideals of $F(\theta)$* is that

$$[\pi/(s_1 + t_1\rho)][\pi/(s_2 + t_2\rho)] \cdots [\pi/(s_n + t_n\rho)]$$
$$= \rho^{\alpha}[\pi/(s_1 + t_1\rho^2)]\ [\pi/(s_2 + t_2\rho^2)] \cdots [\pi/(s_n + t_n\rho^2],$$

where $\alpha = [N(\pi) - 1]\ (e + 2j)/3$ and if $n = 0$ each Eisenstein symbol is taken as unity.

We shall next prove

THEOREM 4. *If p is a rational prime, not a divisor of $6D$, a sufficient condition that p be the product of three conjugate ideals of $F(\theta)$ is that p be a cubic residue of D. If (a) $D = 9$ or $D = q$, a prime, or (b) $F(\theta)$ is of class unity, this condition is also necessary.*

To prove the sufficient condition, suppose that p is a rational prime, not a divisor of $6D$, and a cubic residue of D. Let $p = \pi\pi'$ or $p = \pi$ where π is a primary prime of $F(\rho)$. If D is divisible by 3, p is a cubic residue of 9 and hence is in the form $9u \pm 1$. Then $N(\pi) \equiv 1 \pmod 9$. If D is prime to 3, $j = 0$ and by (c) of Lemma 2, $e = \alpha = 0$. In either case $\rho^{\alpha} = 1$.

Suppose $p = \pi\pi'$. Since it is a cubic residue of D, by (13_2) we have

$$[\pi/(s_i + t_i\rho)]\,[\pi'/(s_i + t_i\rho)] = [\pi/(s_i + t_i\rho^2)][\pi'/(s_i + t_i\rho^2)] = 1$$
$$(i = 1, 2, \cdots, n).$$

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* Bachmann, *Die Lehre von der Kreisteilung*, p. 198.

By (13_1) these equations are equivalent to

$$[\pi/(s_i + t_i\rho)] = [\pi/(s_i + t_i\rho^2)] \quad (i = 1, 2, \cdots, n).$$

If $p = \pi$ we obtain these equations at once from our hypothesis that p is a cubic residue of D. The sufficient condition of the theorem follows from these equations and Theorem 3.

To prove (a), we note that $n = 0$ or 1. We may then retrace our steps in the above sufficiency proof and find that if a prime p is the product of three conjugate ideals of $F(\theta)$ then it is a cubic residue of D.

To prove (b), suppose $F(\theta)$ is of class unity and that $N(\lambda) = p$, where λ is an integer of $F(\theta)$ and p is a prime, not a divisor of $6D$. By the last paragraph in § 2, $\delta^6\sigma^3 p = \psi(\alpha_i)$, where the α_i are rational integers, α_0 and α_1 being divisible by k. If D is prime to 3, by (8) and Theorem 1 it may be shown that a and b are divisible by D. By Lemma 1, k^2 divides a and b, and by (a) of Lemma 2, D contains no square factor. Hence a and b are divisible by kD. Then from (6) and the fact that $\alpha_0 = \alpha_0' k$ and α_1 are divisible by k we find $\delta^6\sigma^3 p \equiv \alpha_0^3 \pmod{k^3 D}$ or

$$\delta^6\sigma'^3 p \equiv \alpha_0'^3 \pmod{D}.$$

By (c) and (e) of Lemma 1, σ' is prime to $D = kD'$, and by hypothesis δ is prime to D. Hence p is a cubic residue of D.

If D is divisible by 3 then $D = 9D_1$, and we may repeat the above argument with D replaced by D_1. It remains to show that p is a cubic residue of 9. From $fD = 3Q - P^2$ we have $P = 0$, $\delta = 1$, $Q = a = 3a'$, $R = b$. By (a) of Lemma 1, σ is prime to 3 and hence by (5), $4a'^3 + b^2 \equiv -3 \pmod 9$. Therefore $a = 3a' \equiv -3$, $b = \pm 1$, (mod 9). Since $\sigma^3 p = \psi(\alpha_i)$, we find from these congruences and (6) that

$$\sigma^3 p \equiv \xi^3 - 3\xi\eta^2 + \eta^3 \pmod 9,$$

where $\xi = \alpha_0 + 2\alpha_2$, $\eta = \mp\alpha_1 - \alpha_2$. For the case $b \equiv 1 \pmod 9$, this congruence is essentially the identity used by Nowlan * who showed that the right member is divisible by 3 or is congruent to ± 1, modulo 9, for every ξ, η. Since the left side is prime to 3, it follows that $p \equiv \pm 1 \pmod 9$, i. e. p is a cubic residue of 9. This completes the proof of the theorem.

The condition of Theorem 4, i. e. that p be a cubic residue of D, is obviously equivalent to the condition that p occur in one of a certain set of arithmetic progressions, $T_i + Dz$, where the T_i are rational integers. If D is prime to 3, the α of Theorem 3 is zero for every p and it may be shown

* Ibid., p. 379.

that the condition of Theorem 3 is equivalent to the condition that p occur in one of a certain set of arithmetic progressions of the above type. If D is divisible by 3, we may obtain a similar result; except that since $\alpha \not\equiv 0$ we have three sets of progressions instead of one, the set which is used in applying the criterion to a given p depending on the residue of p, modulo 9.*

Westlund † treated the problem of determining the degree of the prime ideal factors of a rational prime p in the field defined by a root of

$$\phi(y) = y^3 - 3qy - qA = 0,$$

where q is a prime, $4q = A^2 + 27B^2$, $A \equiv q \equiv 1 \pmod 3$. The discriminant of the field is q^2. He found that if p is a prime, not a divisor of $6qB$, then p is the product of three prime ideals of the field if and only if $\phi(y) \equiv 0 \pmod p$ has a solution. This is implied by Theorem 2. However, Theorem 4 gives a simpler criterion, namely that p be a cubic residue of q.

5. We shall now prove

THEOREM 5. *A necessary condition that $F(\theta)$ be of class unity is that $D = 9$ or $D = q$, a prime.*

Suppose $F(\theta)$ is of class unity and that D contains two prime factors, one of which $\not\equiv 3$. Consider first the case where D is prime to 3. By (a) of Lemma 2, the prime factors of D are distinct. We may then determine integers, λ_i, n_i, of $F(\rho)$ which are prime to D and such that

$$\begin{aligned} [\lambda_1/(s_1 + t_1\rho)] &= [\eta_2/(s_2 + t_2\rho^2)] = [\lambda_2/(s_2 + t_2\rho)]^2 \\ &= [\eta_1/(s_i + t_i\rho^2)] = \rho, \\ [\lambda_i/(s_i + t_i\rho] &= [\eta_i/(s_i + t_i\rho^2)] = 1 \qquad (i = 3, \cdots, n.), \end{aligned}$$

where the last two equations are ignored if $n = 2$. Let δ be an integer of $F(\rho)$ such that

$$\delta \equiv \lambda_i \pmod{s_i + t_i\rho}, \quad \delta \equiv \eta_i \pmod{s_i + t_i\rho^2} \qquad (i = 1, 2, \cdots, n.).$$

After adding proper multiples of D to δ we may assume $\delta \equiv 2 \pmod 9$. Then δ is prime to $9D$. By a theorem due to Fanta ‡ there is an infinitude of primes of $F(\rho)$ in the form $\delta + 9Dz$, where z is an integer of $F(\rho)$.

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* Cf. Bauer, *Archiv der Mathematik und Physik*, (3), Vol. 6 (1903), pp. 221-4; Bulletin, National Research Council, Number 28, Vol. 5 (1923), *Algebraic Numbers*, p. 49.

† *Deutsche Mathematiker Vereinigung*, Vol. 22 (1913), pp. 135-40; Bulletin, National Research Council, *Algebraic Numbers*, pp. 57-8.

‡ *Monatshefte für Mathematische Physik*, Vol. 12 (1901), pp. 1-44; Bulletin National Research Council, *Algebraic Numbers*, p. 21.

Let π be such a prime which does not divide $6f'D$. Since $\pi \equiv \delta \equiv -1$, (mod 3), π is primary. We have

$$[\pi/(s_i + t_i\rho)] = [\lambda_i/(s_i + t_i\rho)], \quad [\pi/(s_i + t_i\rho^2)] = [\eta_i/(s_i + t_i\rho^2)] \qquad (i = 1, 2, \cdots, n).$$

Then by Theorem 3 and the definition of the λ_i, η_i it follows that the rational prime $p = \pi$ or $N(\pi)$ is the product of three conjugate ideals of $F(\theta)$. But

$$\rho = [\pi/(s_1 + t_1\rho)] = [\pi/(s_1 + t_1\rho^2)]^2 = [\pi'/(s_1 + t_1\rho)],$$

and therefore p is a cubic non-residue of D, contrary to (b) of Theorem 4. This proves the theorem if D is prime to 3.

Consider the case where $D = 9D_1 \neq 9$. Let λ_i, η_i be integers of $F(\rho)$ prime to D and such that

$$[\lambda_1/(s_1 + t_1\rho)] = \rho^{e+1}, \quad [\lambda_i/(s_i + t_i\rho)] = [\eta_j/(s_j + t_j\rho^2)] = 1 \qquad (i = 2, \cdots, n;\ j = 1, 2, \cdots, n).$$

Let δ and π be defined as before, except that D is replaced by D_1. Since $\delta \equiv 2$ (mod 9) it follows that $N(\pi) - 1 \equiv 3$ (mod 9) and hence $\rho^a = \rho^{e+1}$. By (c) of Lemma 2, $e \neq 2$. We then obtain as before a contradiction of (b) of Theorem 4. This completes the proof of the theorem.

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On the Problem of Existence of Algebraic Functions of Two Variables Possessing a Given Branch Curve.

By Oscar Zariski.

I. *Introduction and Statement of the Problem.*

1. The problem which we investigate in this paper is a direct generalization of the well-known existence problem in the theory of algebraic functions of one variable to the case of two independent variables, and may be formulated as follows:

Does an algebraic function z of x and y exist, possessing a preassigned curve f as branch curve?

This problem was first considered by F. Enriques in a paper published in 1923.* Enriques finds that the assigned substitutions on the branches of the function z must satisfy certain conditions, which he calls *conditions of invariance.* In conversations, from which this investigation arose, it was pointed out to me by S. Lefschetz that the above problem is intimately bound up with the Poincaré *group* of f relative to its carrying complex projective plane, the conditions given by Enriques merely expressing the fact that the assigned substitutions must satisfy the generating relations of the fundamental group of the curve f. By making use of the theorem of existence of algebraic functions of one variable, possessing given branch points, Enriques proves that these conditions are also sufficient, and hence announces his existence theorem for algebraic functions of two variables.

Enriques' result, important as a principle, is nevertheless incomplete, since we do not know anything about the fundamental group of an algebraic curve. The complete solution of the existence problem depends upon the solution of the following purely topological problem:

Given an algebraic curve, to find its fundamental group.

In this paper we attempt to throw some light upon the structure of the fundamental group. Our investigation is far from giving the complete solution of the proposed problem. However, the theorems established below settle

* "Sulla costruzione delle funzioni algebriche di due variabili possedenti una data curva di diramazione," *Annali di matematica pura ed applicata*, Ser. 4, Vol. 1 (November 1923), pp. 185-198.

the problem in the most simple cases, and suggest a possible direction for a deeper investigation.

2. Riemann announced the following existence-theorem: * There exists an algebraic function $y(x)$ (determined to within a birational transformation), which possesses in the plane of the complex variable x given critical points $a_1, a_2, \cdots, a_m$, and the branches of which—with respect to a given system of consecutive non-intersecting loops Oa_i—are permuted according to assigned substitutions $S_1, S_2, \cdots, S_m$, provided they satisfy the following two conditions:

(a) $S_1 S_2 \cdots S_m = 1$;

(b) the substitutions S_i generate a transitive group.

In the case of two independent variables, we have an algebraic function z of x and y defined by an irreducible algebraic equation

$$\phi(x, y, z) = 0. \tag{1}$$

The *branch curve* f of the function z in the plane of x and y can be defined as the locus, as x varies, of the branch points of z, considered as a function of y alone, by letting $x = \text{const.}$ The equation of f,

$$f(x, y) = 0, \tag{2}$$

is obtained by eliminating z between equation (1) and

$$\partial\phi/\partial z = 0,$$

and by neglecting the *apparent* branch components, which arise from ordinary multiple curves of the surface (1).

If the curve f is given, and if we look for an algebraic function z, which should possess f as branch curve, we must first assign the substitutions on the branches of the hypothetical function z. This is done by considering the *fundamental group*, introduced by Poincaré, for the curve f, which is a two-dimensional Riemann surface immersed in the four-dimensional real space S_4 of the two complex variables x and y, or, in more exact terms: the fundamental group of the residual space $S_4 - f$.

We recall that the fundamental group of f, which we will denote by G, is defined as follows: †

The elements of G are all possible one-dimensional oriented circuits g in

* "Theorie der Abelschen Functionen," §§ 3-5, *Crelle's Journal*, Vol. 54 (1857).

† See, for instance, O. Veblen, "Analysis Situs," pp. 132-133, *The Cambridge Colloquium* 1916, (New York, 1922).

the residual space $S_4 - f$, which start from and end at a given point O. Two circuits are said to be equivalent if one can be carried into the other by a continuous deformation in $S_4 - f$, the point O remaining fixed. Equivalent circuits are not considered as essentially distinct and correspond to one and the same element of the fundamental group. In particular, any circuit g, which can be shrunk into the point O, by a continuous deformation as above, is represented in G by the identity 1, and we write $g = 1$. The product of two elements g_1, g_2 is the circuit obtained by describing first the circuit g_1 and then the circuit g_2. Generally $g_1 g_2 \neq g_2 g_1$. From the definition it follows that g and g^{-1} refer to one and the same circuit with opposite orientations. The choice of the point O has no effect on the structure of the fundamental group.

A finite set of generators of G can easily be constructed. It can be shown,* that any circuit g is equivalent to a circuit g' belonging to a generic "line" l, $\alpha x + \beta y + \gamma = 0$, through O (a two-dimensional manifold, homeomorphic to a sphere). Let $a_1, a_2, \cdots, a_n$ be the intersections of f with l, where n is the order of f. In l, the circuit g' is equivalent to a product of elementary non-intersecting circuits (loops) $g_1, g_2, \cdots, g_n$, which all start at O and surround the points $a_1, a_2, \cdots, a_n$ respectively. Hence the *loops* $g_i (i = 1, 2, \cdots, n)$ *can be taken as generators of* G.

The generators g_i satisfy several relations, called *generating relations.* We will study these relations in the next section. Here it may be observed that evidently

$$g_1 g_2 \cdots g_n = 1, \tag{3}$$

and that other relations are obtained by letting the line l vary in the pencil of lines through O. Any generating relation is of the form:

$$g_1^{k_1} g_2^{k_2} \cdots g_n^{k_n} g_1^{l_1} g_2^{l_2} \cdots g_n^{l_n} \cdots g_1^{m_1} g_2^{m_2} \cdots g_n^{m_n} = 1.$$

Let us write down all the generating relations of G:

$$\phi_1(g_1, g_2, \cdots, g_n) = 1, \quad \phi_2(g_1, g_2, \cdots, g_n) = 1, \cdots, \quad \phi_r(g_1, g_2, \cdots, g_n) = 1. \tag{4}$$

In assigning the substitutions on the branches of an algebraic function z of x and y, it is sufficient to assign the substitutions $s_1, s_2, \cdots, s_n$ relative to the generators g_i. However, these generating substitutions of the monodromie group of the function z cannot be given arbitrarily. By (4), they must satisfy the relations

* S. Lefschetz, L'analysis Situs et la géometrie algébrique, p. 33. Paris, Gauthiers-Villars, 1924.

$$(4') \qquad \phi_1(s_1, s_2, \cdots, s_n) = 1, \qquad \phi_2(s_1, s_2, \cdots, s_n) = 1, \cdots, \phi_r(s_1, s_2, \cdots, s_n) = 1,$$

analogous to (4).

A knowledge of the generating relations (4), and hence of the fundamental group G, is thus essential for the solution of the existence problem, which we are now in a position to formulate in an exact manner. It consists of two parts:

1. *Given an algebraic curve f, to determine its fundamental group.*
2. *Does an algebraic function z of x and y exist, which possesses a given branch curve f, and the values of which are permuted—with respect to the above constructed set of generators g_i—according to given substitutions s_i, satisfying the relations (4').*

The second question is answered in the affirmative by F. Enriques (*loc. cit.*). The possibility of constructing the function, satisfying the above conditions, is, as was shown by Enriques, a consequence of the theorem of existence of algebraic functions of one variable. This paper is hence dedicated exclusively to the investigation of the fundamental group G.

II. *General properties of the group G.*

3. Let

$$f(x, y) = 0$$

be an algebraic curve of order n—irreducible or not. We consider the pencil of lines

$$x = \text{const.}$$

Let $x = a$ be a definite line in this pencil, and let

$$y_i = b_i, \ (i = 1, 2, \cdots, n),$$

be the intersections of the considered line with the curve f. We mark in the plane of the complex variable y the points b_i and we construct a set of non-intersecting loops $g_i = Ob_i$, where by O we mean the point $y = \infty$, center of the considered pencil. As we said in the preceding section, the loops g_i form a set of generators of the group G.

Let x traverse in its plane a closed path, starting from its initial position $x = a$ and returning to it. The set of points b_i will move continuously in the plane y, and will return to its initial position, though the points b_i may be permuted among themselves. The variation of the loops g_i is completely determined by the paths described by the points b_i and by the condition that

they should never intersect. Thus, at the end of the above cyclic variation of the point x, the set of the original loops g_i is transformed into a new set of non-intersecting loops g_i'. More precisely, if the point b_i is carried into b_j then the loop g_i is transformed into some loop g_i' which surrounds the point b_j alone. The two loops g_i' and g_j do not coincide necessarily, but since g_i' surrounds the same point b_j as g_j, g_i' is evidently the transform of g_j by some element g of G, i. e. g_i' and g_j are conjugate elements of G:

$$g_i' = g^{-1} g_j g.$$

The loops g_i and g_i' being equivalent, we obtain the following generating relation:

$$g_i = g^{-1} g_j g.$$

Hence we have the following theorem:

THEOREM 1. *If b_i can be carried into b_j by a cyclic variation of the point x, then the generators g_i and g_j are conjugate elements of the group G.*

If the curve is *irreducible*, then it is always possible to carry any of the points b_i into any other point b_j. Hence, by theorem 1 we deduce:

THEOREM 2. *If the curve f is irreducible, then the n generators g_i are conjugate elements of the group G.*

Let us suppose that the group G is abelian. Then by means of theorem 2, we deduce

$$g_1 = g_2 = \cdots = g_n.$$

The relation $g_1 g_2 \cdots g_n = 1$ becomes in this case

$$g_1^n = 1.$$

On the other hand, no lower power of g_1 is equal to the identity. In fact, if g_1 was of order $n_1 < n$, then the only existing algebraic functions z admitting f as branch curve would be n_1-valued functions (or n_1'-valued, where n_1' is a divisor of n_1), possessing a cyclic monodromie group. But this is impossible, since *any* curve f of order n is the branch curve of an n-valued function, namely of the function z defined by the equation *

$$z^n = f(x, y).$$

Hence, we have the following theorem:

* If $z^n = f(x, y)$, then the branch curve of z is the curve f and the line at infinity of the plane x, y, *unless the order of f is a multiple of n.* In this case the line at infinity does not belong to the branch curve of z.

THEOREM 3. *If the curve f of order n is irreducible, and if it is known that the group G is abelian, then G is necessarily the cyclic group of order n.*

Theorem 3 shows that it is impossible to construct upon an *irreducible* branch curve an algebraic function possessing an abelian non-cyclic monodromie group.

4. *The elementary generating relations.* Let us mark in the plane of the complex variable x the critical points $\alpha_1, \alpha_2, \cdots, \alpha_n$ of the function y. We include among the points α_i also those values of x, which correspond to ordinary multiple points of the curve f (apparent branch points). Let $a\alpha_1, a\alpha_2, \cdots, a\alpha_n$ be a set of non-intersecting loops in the plane x. Under the simplest conditions—but quite general and sufficient for the theory of algebraic surfaces from the point of view of birational transformations—the curve f will only possess ordinary double points and cusps. If the axes of x and y are not specially related to the curve f, then at each critical point α_i two and only two values of y coincide. The investigation of the generating relations which are obtained when x traverses a loop $a\alpha_i$ is due to Enriques. His results can be stated as follows:

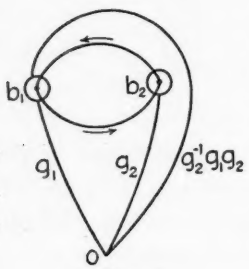

FIG. 1

(a) If α_i corresponds to a simple tangent of the curve f, and if $y_1 = b_1$, $y_2 = b_2$ are the two values of y which tend to coincide when x tends to α_i along the loop $a\alpha_i$, then two corresponding loops g_1, g_2 in the y-plane can be constructed, such that, when x traverses the loop $a\alpha_i$, g_1 is transformed into g_2 and g_2 is transformed into $g_2^{-1}g_1g_2$, while the remaining $n-2$ loops remain invariant (see fig. 1).

This leads to the generating relation:

$$g_1 = g_2. \tag{5}$$

(b) A critical value α_i, which corresponds to an ordinary double point, can be thought of as being the limit of two simple branch points α_i', α_i'', at which the same two branches y_1 and y_2 are permuted and which come together. The loop $a\alpha_i$ is the limit of a circuit, which surrounds the infinitely near branch points α_i', α_i'' and which produces the identical substitution $(b_1b_2)^2$. Therefore, the transformation of the loops in the y-plane,

which arises when x traverses the loop $a\alpha_i$, is the square of the transformation mentioned above with reference to a simple tangent, i. e., the following:

$$g_1' = g_2^{-1}g_1g_2, \qquad g_2' = g_2^{-1}g_1^{-1}g_2g_1g_2.$$

The corresponding generating relation in this case is the following:

$$g_1g_2 = g_2g_1. \tag{6}$$

(c) Finally, if α_i corresponds to a cusp, it is the limit of three simple branch points, and the resulting transformation of the loops must be the cube of the transformation mentioned above, i. e., the following:

$$g_1' = g_2^{-1}g_1^{-1}g_2g_1g_2, \qquad g_2' = g_2^{-1}g_1^{-1}g_2^{-1}g_1g_2g_1g_2.$$

In this case the corresponding generating relation becomes:

$$g_1g_2g_1 = g_2g_1g_2. \tag{7}$$

Remark. The above transformation of the loops can also be deduced rigorously (see Enriques, quoted paper), by a careful inspection of the expansion of the function y in the three considered cases.

The above considerations show that when a curve acquires an ordinary double point, an equality (5) between two generators is replaced by a permutability relation (6). The appearance of a cusp destroys a permutability relation, and replaces it by a more complicated relation (7). The effect of the singularities of an algebraic curve upon its fundamental group is thus explicitly revealed.

We will call the above generating relations (5), (6) and (7), which correspond to the loops $a\alpha_i$, *the elementary generating relations.*

The form of the above stated elementary generating relations is closely connected with the topological structure of the neighborhood of a multiple point of an algebraic curve. To such a multiple point there corresponds a closed curve, or a set of closed curves, in the ordinary 3-dimensional space, by means of the following construction: Let us consider a neighborhood (a 4-cell) of a multiple point P. The boundary S of the 4-cell (3-dimensional sphere) will meet the curve f along a closed one-dimensional circuit, or along a set of such circuits, l. By a stereographic projection of S on to the ordinary 3-dimensional space the circuit l is projected into a closed curve l', or into a set l' of such curves, in that space. If P is a simple point of f, then l' is a single unknotted curve. If P is an ordinary double point of f, then l' is

composed of two unknotted linked curves.* If P is a cusp, then l' is the well-known trefoil knot. The above elementary relations are merely the generating relations of the fundamental group of the curve l'.

The following theorem is an implicit consequence of the existence theorem, as it is stated by Enriques:

THEOREM 4. *The elementary generating relations together with the relation* $g_1 g_2 \cdots g_n = 1$, *form a complete set of generating relations,* i. e. *every relation between the generators is a consequence of them.*

In fact, Enriques shows that there always exists an algebraic function z, if the substitutions on the branches of z satisfy the elementary generating relations and the relation (3) of section 2. This proves the non-existence of other independent relations between the generators.

5. *The fundamental group of a curve* $\bar{f}$, *limit of a variable curve* f. An important consequence of the above considerations concerns the fundamental group of a curve $\bar{f}$ which is the limit of a variable curve f. If $\bar{f}$ has the same singularities as f, then f and $\bar{f}$ are isotopic, and hence possess the same fundamental group. Suppose now that when f tends to $\bar{f}$, a new multiple point is acquired. For simplicity, let us suppose that when f tends to $\bar{f}$, two simple critical points α_1 and α_2 of the variable function y come together, so that $\bar{f}$ acquires a new ordinary double point. What happens to the generating relations? It is evident that any elementary generating relation, which corresponds to a critical point α_j different from α_1 and α_2, will remain a true relation between the generators of the fundamental group of the limit curve $\bar{f}$. The only relations which are destroyed are those relative to the critical points α_1 and α_2, namely the one relation $g_1 = g_2$, since at the limit, when α_1 and α_2 come together at a point α, the two loops g_1 and g_2 pass through α. However, we obtain the relation $g_1 g_2 = g_2 g_1$ between the generators of the group G of the variable curve f, when x traverses a closed path surrounding both critical points α_1 and α_2, and this relation is *not* destroyed, when α_1 and α_2 coincide at α: it becomes the elementary generating relation corresponding to the loop $a\alpha$. Thus in this case, one generating relation ($g_1 = g_2$) is destroyed, and is replaced by the relation $g_1 g_2 = g_2 g_1$, which is a *consequence* of the old relation.

This conclusion holds in the most general case, of any acquired new

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* This is a consequence of a theorem of the Kronecker index of two algebraic curves on a surface, Lefschetz, *loc. cit.*, p. 19.

multiple point, namely: if several critical points $\alpha_1, \alpha_2, \cdots, \alpha_r$ come together to form a new singularity, or a singularity of higher order, of the limit curve $\bar{f}$, then r corresponding elementary relations are destroyed, and are replaced by *one* elementary relation, which on the variable curve f corresponds to a closed path surrounding all the r mentioned points α, and which is therefore a consequence of the destroyed r relations. We have hence the following theorem:

THEOREM 5. *If a variable curve f tends to a limit curve $\bar{f}$, then several relations between the generators of the group G of f are abolished, but no new relation arises which is not a consequence of the old relations.*

We may express this theorem also in the following form:

All the generating relations of the fundamental group of the limit curve are also generating relations of the fundamental group of the variable curve.

This last theorem still holds, even if the limit curve $\bar{f}$ degenerates, provided that all its components are simple. If, however, $\bar{f}$ possesses multiple components (so that the number of acquired singularities is infinite), the above considerations cannot be applied without modification.

We see that the fundamental group of the limit curve is simply more general than the fundamental group of the variable curve. Hence, the more numerous and the more complicated the singularities of an algebraic curve f, the larger the class of birationally distinct algebraic functions which can be constructed upon f as branch curve.

6. *The continuous system* $\{f\}$. Let f_0 be a definite curve of order n with assigned singularities, and let us consider a complete irreducible continuous system of curves f of order n, possessing the same singularities as f_0, to which f_0 belongs. We leave aside the question, as yet unsolved in general, whether the system is unique.* If f_0 belongs to several continuous systems, then we consider one definite system. We denote it by $\{f\}$. The *general* curve f of $\{f\}$ is isotopic to f_0 and hence possesses the same fundamental group

* This question is solved for the case in which the assigned singularities are ordinary double points. It is even proved that *all curves of given order and genus form an irreducible continuous system.*

See F. Enriques and O. Chisini, *Courbes et fonctions algébriques d'une variable,* Chap. III, p. 365 (Paris, Gauthier-Villars, 1926); F. Severi, *Vorlesungen über algebraische Geometrie,* Anhang F., pp. 342-349 (Leipzig, Teubner, 1921).

as f_0. We consider again a definite line l, say $x = a$, and a set of loops $g_1, g_2, \cdots, g_n$ in the y-plane. Instead of keeping the curve f fixed and letting the line l vary in a pencil, as we did before, we keep the line l fixed and we let the curve f vary in the continuous system $\{f\}$, starting from and returning to its initial position f_0. Let

$$\phi(g_1, g_2, \cdots, g_n) = 1 \tag{8}$$

be a definite generating relation of the fundamental group of the original curve f_0. When f varies in $\{f\}$ and returns to its original position f_0, the original loops g_i will be transformed into new loops g_i'. The relation (8), with which we started, will become

$$\phi(g_1', g_2', \cdots, g_n') = 1, \tag{8'}$$

and this is necessarily a true relation between the old generators g_i, possibly different from that with which we started. Thus we have a means of obtaining several new generating relations, by transforming a given relation in the above mentioned way.

Let us make the following hypotheses on the continuous system $\{f\}$.

(a). *The general curve f of $\{f\}$ cuts out on a generic line l a general set of n points, i. e. the dimension of the series cut out by the curves of the system $\{f\}$ on l is exactly n.*

(b). *Those curves of $\{f\}$ which intersect the line l in n given points still form an irreducible continuous system.*

Under those conditions it can be proved that the fundamental group of the general curve of the system $\{f\}$ is necessarily cyclic of order n.

Indeed, by hypothesis (a), we can assign in advance arbitrary paths for the intersections b_i $(i = 1, 2, \cdots, n)$ of the variable curve f of $\{f\}$ with the line l. We start with the intersections b_i^0 of the original curve f_0 with l, and we consider an arbitrary set of paths which carries the set of points b_i^0 into its original position, the points b_i^0 being possibly permuted. The continuous succession of the intermediate positions of the variable curve f, determined (to within a certain degree of freedom, which can be eliminated by putting further linear conditions on the coefficients of f) by the intermediate positions of the points b_i, may carry us possibly to a curve $\bar{f}_0$, different from the original curve f_0. But if hypothesis (b) holds, then it is possible to

carry $\bar{f}_0$ into f_0, keeping the intersections b_i^0 fixed. In conclusion, we have that it is always possible to find a cyclic variation of the curve f, starting from f_0, such that when f returns to its initial position f_0 the points b_i have traversed an arbitrary set of paths, given in advance. Since the most general transformation of the loops g_i can be obtained by letting the set of points b_i move along conveniently chosen paths, which carry the set back to its original position, we deduce that if $\phi(g_1, g_2, \cdots, g_n) = 1$ is a generating relation, then *any* transformation of the loops can be applied to ϕ in order to obtain a new generating relation.

Let us take in particular an elementary generating relation $g_1 = g_2$, corresponding to a tangent line to the curve f_0, and let us consider the following elementary transformation T_i $(i = 1, 2, \cdots, n-1)$ of the loops:

$$(9) \qquad T_i: g_i' = g_{i+1}; \quad g'_{i+1} = g_{i+1}^{-1} g_i g_{i+1}; \quad g_j' = g_j \quad (j \neq i, i+1).$$

The transformation T_i is obtained by interchanging b_i^0 and $b^0{}_{i+1}$ along conveniently chosen paths (see Fig. 1, sec. 4). The transformation T_2, applied to the relation $g_1 = g_2$, leads to the relation $g_1 = g_3$. If we apply to this relation the transformation T_3, we obtain $g_1 = g_4$. By applying successively the transformations T_i, we obtain

$$g_1 = g_2 = \cdots = g_n.$$

Hence the group G is cyclic, and due to relation (3) it is necessarily of order n, q. e. d.

Hence, we have the following theorem:

THEOREM 6. *If the general curve of an irreducible complete continuous system $\{f\}$ possesses a non-cyclic fundamental group, then at least one of the following two conditions must be satisfied:* (a) *either the curves of the system cut out on a generic line special sets of n points* (for which, say, some invariant vanishes), (b) *or the subsystem of curves of $\{f\}$, which pass through n given points on a line, breaks up into several irreducible continuous systems.*

Thus the study of the series of sets of points cut out by the curves of $\{f\}$ on a generic line of the plane, is of prime importance for the determination of the fundamental group of the general curve f of the system. In several cases which we study below, the knowledge of this series leads directly to the determination of the group.

III. *Investigation of Some Important Particular Cases.*

7. *The fundamental group of a curve possessing ordinary double points only.* In this section we prove the following theorem:

THEOREM 7. *The fundamental group of an irreducible curve f of order n, possessing ordinary double points only, is cyclic of order n.*

First we consider separately the case in which the curve f has no singularities at all. Since two curves of the same order and without singularities are isotopic and hence have one and the same fundamental group, it is enough to prove the theorem for some special curve of order n. We consider a curve of order n, which possesses a flex of order $n-2$, at which the n intersections of the flex tangent with the curve coincide. For instance, we may consider the curve

$$y^n = (x+1)^n - 1,$$

which has no singularities at all, and which possesses a flex of order $n-2$ at the origin.

The development of y in the neighborhood of $x=0$ starts with the term $(nx)^{1/n}$. It follows that when x traverses in its plane a small circle of center $x=0$, the n values of y: y_1, $y_2 = y_1 \cdot e^{2\pi i/n}$, $y_3 = y_2 \cdot e^{2\pi i/n}$, $\cdots$, $y_n = y_{n-1} \cdot e^{2\pi i/n}$, move along a simple closed curve C surrounding the point $y=0$, and that the argument of each y_i is increased by $2\pi/n$. We start with the initial value $x=-1$, and we construct in the y-plane a set of loops g_i, in the way indicated by the figure 2 (the point O, being the point at infinity, must be taken outside the curve C). (Fig. 2.) When x moves along the circle $|x|=1$ in the positive direction and comes back to the initial value $x=-1$ the loops g_i are transformed as follows:

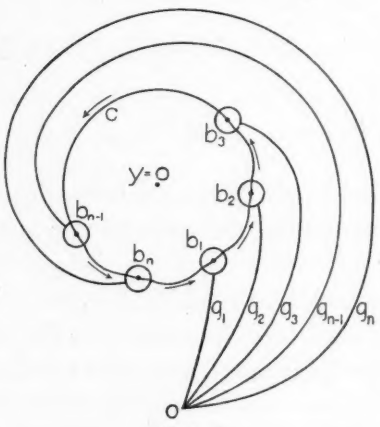

FIG. 2

$$g_1' = g_2; \qquad g_2' = g_3; \quad \cdots; \qquad g'_{n-1} = g_n;$$
$$g_n' = g_n^{-1} g_{n-1}^{-1} \cdots g_2^{-1} g_1 g_2 \cdots g_{n-1} g_n.$$

It follows immediately that

$$g_1 = g_2 = g_3 = \cdots = g_n,$$

hence the group G is cyclic. Relation (3) shows that it is cyclic of order n.

Since the preceding proof is based exclusively upon the fact that the general curve of order n can acquire a flex of order $n-2$ at a simple point of the curve, it applies to every irreducible curve possessing this property, no matter what are the singularities of the curve. It is even sufficient to know that a given curve possesses (or can acquire, by varying in a continuous system) a flex of order $n-3$, in order to deduce that its fundamental group is cyclic of order n. Indeed, if a curve f possesses a flex of order $n-3$, at which $n-1$ intersections with the tangent coincide, then, by exactly the same reasoning as above, we can show that the corresponding $n-1$ generators $g_1, g_2, \cdots, g_{n-1}$ are equal. Relation (3) shows that $g_n = g_1^{-(n-1)}$, hence the group is cyclic. By theorem 3, sec. 3, it follows then that the order of the group G is n. If the curve f itself does not possess a flex of order $n-3$, but if it can be deformed into a curve $\bar{f}$ possessing a flex of this order, then, since the fundamental group of $\bar{f}$ is cyclic of order n, we deduce, by theorem 5, sec. 5, that the group G of f is also cyclic of order n. Applying this result to curves of order 3 and 4 we have the following theorem:

THEOREM 8. *The fundamental group of any irreducible cubic or quartic curve is cyclic of order* 3 *and* 4 *respectively, with the only exception of the three-cuspidal quartic curve.*

The three-cuspidal quartic does not possess flexes, and in fact its fundamental group is not cyclic, as we will show later.

To prove the theorem 7, we prove first the following lemma:

LEMMA. *The fundamental group of n lines in arbitrary position is abelian.*

Let $L_1, L_2, \cdots, L_n$ be n lines in arbitrary position. Let be: A the intersection of L_1 and L_2, l a line through a fixed point O, very near to the line OA, and b_i the intersection of L_i and l. When l tends to the line OA, the two intersections b_1 and b_2 tend to coincide at the *double point* A of the curve $L = L_1 + L_2 + \cdots + L_n$. Hence in l a set of generators $g_1, g_2, \cdots, g_n$ can be constructed, such that the following relation holds:

$$g_1 g_2 = g_2 g_1. \tag{10}$$

Now, let the curve L vary and come back to its initial position, the lines L_i being possibly permuted. Since the L_i's are arbitrary lines, we may arrange

the variation in such a way that the intersections b_i with the fixed line l shall follow arbitrarily prescribed paths. Hence (see sec. 6) any transformation of the loops, if applied to (10), will lead to a true relation. We apply successively the transformations T_i, defined by the formulas (9) of sec. 6, and we obtain:

$$(11) \qquad g_1g_2 = g_2g_1; \qquad g_1g_3 = g_3g_1; \cdots; \qquad g_1g_n = g_ng_1.$$

Transforming each of the relations (11) by T_1 we obtain:

$$(12) \qquad g_2g_1 = g_1g_2; \qquad g_2g_3 = g_3g_2; \cdots; \qquad g_2g_n = g_ng_2.$$

Transforming each of the relations (12) by T_2 we obtain:

$$g_3g_1 = g_1g_3; \qquad g_3g_2 = g_2g_3; \cdots; \qquad g_3g_n = g_ng_3.$$

Transforming successively by $T_3, T_4, \cdots, T_{n-1}$, we deduce that any two generators are permutable, and hence that the group G is abelian.

Now let f be an irreducible curve of order n with ordinary double points only. The continuous system $\{f\}$, in this case uniquely determined (see footnote on p. 313), contains in particular curves which degenerate into n arbitrary lines.* Now, if f, varying continuously, degenerates into n lines, then the fundamental group becomes abelian, by the Lemma proved above. But in this case, the group G must originally have been abelian, since by theorem 5, sec. 5, the generating relations of the group of the limit curve (in this case: $g_ig_j = g_jg_i$) hold also for the generators of the group of the variable curve. Hence the fundamental group of f is abelian, and by the theorem 3 sec. 3 it is then necessarily the cyclic group of order n.

The previous considerations are applicable without modification to a reducible curve f of order n, which breaks up into s simple irreducible curves $f_1, f_2 \cdots f_s$, possessing ordinary double points only and having simple mutual intersections only. It is still possible to let f degenerate into n lines, and hence the fundamental group of f is still abelian. *But it is not cyclic any more.* Let n_i be the order of f_i $(n = n_1 + n_2 + \cdots + n_s)$, and let g_{r_i}[4]

* This follows from a noted principle, announced by F. Enriques in 1904, "Sulla proprietà caratteristica delle superficie algebriche irregolari," *Rendiconto delle sessioni della R. Accademia delle Science dell'Instituto di Bologna*, nuova serie, Vol. 9 (1904-1905), pp. 5-13, and completed later by F. Severi, *Vorlesungen über algebraische Geometrie:* Anhang F, No. 7. Leipzig, Teubner, 1921. In that part which interests us, this principle says that: *a reducible curve of order n, composed of s simple irreducible curves of genus $p_1, p_2, \cdots, p_s$ respectively, can always be considered as the limit of an irreducible curve of the same order n and of genus p, provided that* $p \geqq \Sigma p_i$. In our case each p_i is $= 0$, hence the general irreducible curve of given genus p and order n can always be degenerated into n lines.

$(i = 1, 2, \cdots, s;\ r_i = 1, 2, \cdots, n_i)$ be a set of non-intersecting loops, the loops $g_1^{(i)}, g_2^{(i)}, \cdots, g_{n_i}^{(i)}$ corresponding to the intersections of the component f_i with a fixed line l. The curve f_i, being irreducible, it follows (theorem 1, sec. 3) that the generators $g_{r_i}^{(i)}$, i being fixed, are conjugate elements of the group G, and since G is abelian, it follows that

$$g_1^{(i)} = g_2^{(i)} = \cdots = g_{n_i}^{(i)}. \qquad (i = 1, 2, \cdots, s).$$

Thus the group G is generated by s generators $g_1, g_2, \cdots, g_s$, where $g_i = g_1^{(i)} = g_2^{(i)} = \cdots = g_{n_i}^{(i)}$. Relation (3) becomes in this case

$$g_1^{n_1} g_2^{n_2} \cdots g_s^{n_s} = 1. \tag{13}$$

Hence we have the theorem:

THEOREM 9. *The fundamental group of a reducible curve f, possessing ordinary double points only and composed of s irreducible curves $f_1, f_2, \cdots, f_s$ of orders $n_1, n_2, \cdots, n_s$ respectively, is abelian, and is generated by s generators $g_1, g_2, \cdots, g_s$, connected by one generating relation* (13).

By introducing a new set of generators the relation (13) can be simplified as follows: It is always possible * to reduce the matrix

$$\| n_1, n_2, \cdots, n_s \|$$

to its normal form:

$$\| t, 0, \cdots, 0 \|,$$

where t is the H. C. F. of $n_1, n_2, \cdots, n_s$, by a unimodular linear integral transformation of the generators. For the new set of generators, which we still denote by g_i, the relation (13) becomes simply:

$$g_1^t = 1.$$

It is thus seen that *G is an abelian group, generated by $s - 1$ independent elements and by one element of order t.*

Corollary: The monodromie group of an algebraic function z constructed upon f as branch curve must be abelian and must be generated by one substitution of order t (or a divisor of t) and by $s - 1$ other arbitrary substitutions. The order of the function z can be arbitrarily great, except when $s = 1$.

8. *The fundamental group of the branch curve of the general cubic surface.*

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* Frobenius, "Theorie der linearen Formen," *Journal für Mathematik*, Vol. 86 (1879).

Let

$$F(x, y, z) = 0 \tag{14}$$

be the equation of the cubic surface. The branch curve f of the function z is a sextic with 6 cusps on a conic, the equation of which has the following form:

$$f(x, y) = \phi_2^3(x, y) + \phi_3^2(x, y) = 0, \tag{15}$$

where $\phi_2 = 0$ and $\phi_3 = 0$ represent a conic and a cubic curve respectively. By letting ϕ_2 and ϕ_3 vary, we obtain a continuous system $\{f\}$, ∞^{15}, of sextic curves, possessing 6 cusps on a conic. The condition that a plane sextic curve should possess a cusp in non-assigned position is expressed by two relations between the coefficients of the curve. If we admit, that the 12 relations, which express the condition that a sextic curve should possess 6 cusps in non-assigned position, are *independent,* then it follows that these sextic curves form a system ∞^{15}. The above system $\{f\}$ is hence complete. But the curves of this system possess 6 cusps in particular position: they lie on a conic. Hence it seems that there must exist a second continuous system, also ∞^{15}, of sextics having 6 cusps not on a conic, since it is highly improbable that 6 cusps of an irreducible sextic *must* lie on a conic.

We have raised this question (quite unessential for the further investigation) only because of the opportunity which it offers to raise and solve another important question, concerning the fundamental groups of two curves having the same singularities. Let us suppose that there exist two complete continuous systems, both ∞^{15}, of sextics possessing 6 cusps. Then the question arises, whether the fundamental group of a sextic f whose 6 cusps are on a conic, is the same as the fundamental group of a sextic ψ, whose 6 cusps are not on a conic. They would be the same, of course, if it were possible to pass from f to ψ by an isotopic deformation. But such a deformation is *impossible,* at least in the algebraic field, i. e. if the intermediate positions of the deformed curve are *algebraic* curves. In fact, to pass from f to ψ it is necessary to pass through a curve $\bar{f}(=\bar{\psi})$ common to both systems $\{f\}$ and $\{\psi\}$. If the deformation is isotopic, then $\bar{f}$ must not possess new acquired singularities. This is impossible, since it can be easily proved that a sextic with only 6 cusps cannot belong to both systems $\{f\}$ and $\{\psi\}$.

Now, to cut short the discussion, we prove that *the fundamental groups of the above two types of sextics are necessarily distinct* (*non-isomorphic*).

In fact, let us suppose for a moment that the fundamental group of the sextic ψ is the same as the fundamental group of f. Under this hypothesis, any set of generating substitutions of the monodromie group of an algebraic function z, possessing f as branch curve, will also correspond to an actual

function Z, of the same order as z, having ψ as branch curve. In particular, since f is the branch curve of a three-valued function, defined by equation (14), it is possible to construct a three-valued function Z, defined by an equation of the third degree in Z,

$$\bar{F}(x, y, Z) = 0, \tag{16}$$

and possessing ψ as simple branch curve. The surface $\bar{F}$ possesses a net of elliptic curves C, cut out by the planes $\alpha x + \beta y + \gamma = 0$. Hence, by a well known theorem of Castelnuovo-Enriques,* $\bar{F}$ is a rational surface or a ruled surface. Now, $\bar{F}$ cannot be a ruled surface of genus $p > 0$, because it must be a regular surface. Indeed, the irregularity of an algebraic surface depends upon the following elements only: the order of the branch curve, its fundamental group and the assigned substitutions. These elements are the same for the surface $\bar{F}$ as for the cubic surface F. Since F is regular, it follows that $\bar{F}$ is also regular, hence rational. By another well-known theorem on the reduction of complete linear systems of elliptic plane curves,† it is possible to transform birationally the *rational* surface $\bar{F}$ into a plane in such a way, that the curves C of the above net (which is of the degree 3) be transformed into cubics through 6 points. By mapping the plane on a cubic surface

$$F(x', y', z') = 0,$$

whose ∞^3 plane sections correspond to the ∞^3 cubics through the above 6 points, we will have a birational correspondence between the two surfaces $\bar{F}$ and F which transforms the plane sections

$$\alpha x + \beta y + \gamma = 0$$

of $\bar{F}$ into the plane sections of F, cut out by the planes through a point, which we may suppose to be the planes

$$\alpha' x' + \beta' y' + \gamma' = 0.$$

It follows that the branch sextic ψ of the function Z is transformed by a collineation into the branch sextic f of the function z', which is impossible, since the 6 cusps of f lie on a conic.

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* G. Castelnuovo and F. Enriques, "Sopra alcune questioni fondamentali nella teoria delle superficie algebriche," *Annali di Matematica pura ed applicata*, Vol. 6 (series 3), 1900, sec. 17.

† See, for instance, F. Enriques and O. Chisini, *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche*, Vol. III, p. 195 (Bologna, N. Zanichelli). French edition (quoted above), Chap. II, p. 281.

Thus we see, that the fundamental groups of two curves having the same order and the same singularities may well be distinct, provided they belong to two distinct complete continuous systems.

We pass now to the determination of the fundamental group of the sextic f, given by the equation (15).

The set Γ_6 of the intersections of f with an arbitrary line, say with the line $y = 0$, is given by the roots of the polynomial

$$f_6(x) = f(x, 0) = \phi_2^3(x, 0) + \phi_3^2(x, 0).$$

If we assign the set Γ_6 and if we look for the sextic curves f which pass through Γ_6, we are led in the first place to the problem of representing a given polynomial f_6 of 6th degree as a sum of a square and a cube. This problem was treated by Clebsch,* who showed its connection with the problem of the trisection of the periods in the case of the hyperelliptic curve of genus 2

$$y^2 = f_6(x).$$

Clebsch proves that every polynomial f_6 admits 40 different representations of the above type, and that each of the 40 representations corresponds to two third-period characteristics $(\epsilon)_3$, $(-\epsilon)_3$. From this it follows that: 1) there always exist an infinite number of irreducible sextic curves f in the continuous system $\{f\}$, which pass through a given Γ_6 on the line $y = 0$, and that in assigning Γ_6 we only fix the intersections of the variable conic ϕ_2 and cubic ϕ_3 with $y = 0$; 2) since the intersections of ϕ_2 and ϕ_3 with $y = 0$, when Γ_6 is given, can be determined in 40 different ways, the sextic curves f which pass through Γ_6 belong to 40 distinct continuous systems. Thus hypothesis a) of sec. 6 holds in this case, while hypothesis b) does not hold.

The 40 above continuous systems are correlated to the 40 above mentioned pairs of period-characteristics. It is clear that, if we start with a given sextic curve f, belonging to one of these systems, and let f vary in $\{f\}$, until we come back to a curve $\bar{f}$ belonging to the *same* system, while Γ_6 returns to its initial position, then the transformation of the periods, produced by this cyclic variation of the set Γ_6, must leave *invariant* the pair of the corresponding period-characteristics. Conversely it is evident that any circulation of the Γ_6 which leaves invariant a given pair $(\epsilon)_3$ and $(-\epsilon)_3$ of period-characteristics will always carry a curve f of the corresponding system into a curve $\bar{f}$ of the same system, and then $\bar{f}$ can be carried into f by keeping the Γ_6 fixed. We conclude that it is permissible to transform any generating relation of the fundamental

* *Zur Theorie der binären Formen sechster Ordnung*, 1869.

group of f by those and only those transformations of the loops g_i $(i = 1, 2, \cdots, 6)$, which leave invariant the corresponding pair of period-characteristics.

We write below the formulas for the loop transformations and for the corresponding transformations of the period-characteristics in the hyperelliptic case of any genus p:

$$y^2 = (x - a_1)(x - a_2) \cdots (x - a_{2p+2}). \tag{17}$$

Let L be a simple closed curve passing through the points a_i, and let 0 be a point inside the region Q bounded by L. We construct a set of non-intersecting loops $g_i = 0a_i$, meeting L at the respective points a_i alone. As in section 4, we denote by T_i that transformation of the loops which arises when a_i is carried into a_{i+1} along a path situated in the interior of Q and a_{i+1} is carried into a_i along a path situated in the exterior of Q. Then, for $i = 1, 2, \cdots, 2p + 1$, we have:

$$T_i : g'_i = g_{i+1};\ g'_{i+1} = g_{i+1}^{-1} g_i g_{i+1};\ g'_j = g_j,\ (j \neq i,\ i+1).$$

We construct the two-sheeted Riemann surface R of the curve (17) by joining the sheets along the $p + 1$ cuts: $a_1a_2;\ a_3a_4; \cdots;\ a_{2p+1}a_{2p+2}$, made along the corresponding arcs of the curve L. We fix then a set of $2p$ circuits b_i, a_i on R $(i = 1, 2, \cdots, p)$ in the usual manner: let b_i be a circuit in the upper sheet surrounding the cut $a_{2i-1}a_{2i}$ alone, and let a_i be a circuit which crosses the cuts $a_{2i-1}a_{2i}$, $a_{2p+1}\,a_{2p+2}$ alone. Then, on R,

$$b_i = g_{2i-1} - g_{2i};\ a_i = g_{2i} - g_{2i+1} + g_{2i+2} - \cdots + g_{2p} - g_{2p+1}. \tag{18}$$

Let

$$|| 0, 0, \cdots, 0, \pi i, 0, \cdots 0;\ a_{j1}, a_{j2}, \cdots, a_{jp} ||, a_{ij} = a_{ji},$$

be the normalized period matrix defined by the chosen set of circuits b_i and a_i. Let $g_1, g_2, \cdots, g_p; h_1, h_2, \cdots, h_p$ * be the elements of a period characteristic $(g; h)$. We denote by R_i and Q_i the transformation of the circuits b_i, a_i and the transformation of the characteristic $(g; h)$ respectively, which are involved by the transformation T_i of the loops. We have then, by means of the formula (18), the following formulas for the transformations R_i:

$$R_{2i-1}: \quad (i = 1, 2, \cdots, p) \quad \begin{cases} b_k' = b_k, & (k = 1, 2, \cdots, p); \\ a_i' = a_i - b_i;\ \ a_k' = a_k, & (k \neq i). \end{cases}$$

$$R_{2i}: \quad (i = 1, 2, \cdots, p-1) \quad \begin{cases} b_i' = b_i + a_i - a_{i+1};\ \ b'_{i+1} = b_{i+1} - a_i + a_{i+1}; \\ \qquad\qquad b_k' = b_k \quad (k \neq i, i+1); \\ a_k' = a_k, \qquad\qquad (k = 1, 2, \cdots, p). \end{cases}$$

* The reader will not confuse the elements g of the characteristic with the generators g of the fundamental group.

$$R_{2p}: \begin{cases} b_p' = b_p + a_p;\ b'_k = b_k, & (k \neq p); \\ a_k' = a_k, & (k = 1, 2, \cdots, p). \end{cases}$$

$$R_{2p+1}: \begin{cases} b_k' = b_k; & \\ a_k' = a_k - b_1 - b_2 - \cdots - b_p. & (k = 1, 2, \cdots, p). \end{cases}$$

Consequently, the corresponding transformations Q_i of the period-characteristics are: *

$$Q_{2i-1}: \quad (i = 1, 2, \cdots, p) \quad \begin{cases} g_k' = g_k, & (k = 1, 2, \cdots, p); \\ h_i' = h_i - g_i;\ h_k' = h_k, & (k \neq i). \end{cases}$$

$$Q_{2i}: \quad (i = 1, 2, \cdots, p-1) \quad \begin{cases} g_i' = g_i + h_i - h_{i+1};\ g'_{i+1} = g_{i+1} - h_i + h_{i+1}; \\ \qquad\qquad\qquad\qquad\qquad\qquad g_k' = g_k, \quad (k \neq i, i+1); \\ h_k' = h_k, \quad (k = 1, 2, \cdots, p-1). \end{cases}$$

$$Q_{2p}: \begin{cases} g_p' = g_p + h_p;\ g_k' = g_k, & (k \neq p); \\ h_k' = h_k, & (k = 1, 2, \cdots, p). \end{cases}$$

$$Q_{2p+1}: \begin{cases} g_k' = g_k; & (k = 1, 2, \cdots, p). \\ h_k' = h_k - g_1 - g_2 - \cdots - g_p. & \end{cases}$$

We start with a generating relation corresponding to a simple tangent to the sextic curve f. This relation establishes the equality between two generators, and we may suppose that the equal generators are g_1 and g_2:

$$g_1 = g_2. \tag{19}$$

Moreover it is unessential with what period-characteristic we start. We may suppose that the curve f corresponds to the characteristic:

$$(0, 0; 1/3, 1/3). \tag{20}$$

However, it must be pointed out explicitly that the generating relation (19) implies that the points a_1 and a_2 can be interchanged by means of the transformation T_1. Hence a condition upon the characteristic is introduced, namely, that it be invariant under Q_1. The above chosen characteristic (20) satisfies this condition.

The characteristic (20) is left invariant by the transformations Q_2, Q_3

* See A. Krazer, *Lehrbuch der Thetafunktionen*, Leipzig, Teubner, 1903.

and Q_5. Transforming the relation (19) by the transformation T_2 and then by T_3 we obtain

$$g_1 = g_2 = g_3 = g_4.$$

Since the cube of Q_4 leaves any characteristic invariant modulo 3, we transform the relation $g_1 = g_4$ by the cube of T_4 and we obtain the following relation:

$$g_1 g_5 g_1 = g_5 g_1 g_5. \tag{21}$$

Finally we transform the last relation by T_5 and we obtain

$$g_1 g_6 g_1 = g_6 g_1 g_6. \tag{22}$$

The relation $g_1 g_2 \cdots g_6 = 1$ becomes in this case

$$g_1^4 g_5 g_6 = 1.$$

Hence $g_6 = g_5^{-1} g_1^{-4}$, and substituting in (22), we obtain, by making use of (21) after simple transformations, the following relation:

$$(g_1 g_5 g_1)^2 = 1. \tag{23}$$

Since, by (21)

$$g_1 g_5 g_1 \cdot g_1 g_5 g_1 = g_1 g_5 g_1 \cdot g_5 g_1 g_5 = (g_1 g_5)^3,$$

it follows that

$$(g_1 g_5)^3 = 1. \tag{24}$$

Hence (23) and (24) are the generating relations of the group G, which is generated by the two elements g_1 and g_5. We may take $u = g_1 g_5 g_1$, $v = g_1 g_5$, as new generators, and we can state our result as follows:

The fundamental group of a sextic curve f possessing 6 cusps on a conic (branch curve of the general cubic surface) is generated by two elements of orders 2 and 3 respectively.

9. *The fundamental group of the three-cuspidal quartic.* If the cubic surface, considered above, acquires a double line, then the branch sextic curve degenerates into a line $\lambda_1 = 0$ counted twice and into a three-cuspidal quartic,

$$f_4(x, y) = 0. \tag{25}$$

The last one is of special interest, inasmuch as it is the irreducible curve of lowest order, whose fundamental group is not cyclic (see Theorem 8, sec. 7).

The set Γ_6 of 6 points on the x-axis is made up in this case of a double

point, say $x = \xi$, and of 4 simple points—intersections of $y = 0$ with the quartic (25). We have then

$$f(x, 0) = f_4(x, 0)(x - \xi)^2 = \phi_2^3(x, 0) + \phi_3^2(x, 0). \tag{26}$$

Here, in general, i. e. for an arbitrary choice of the line $y = 0$, $\phi_2(x, 0)$ and $\phi_3(x, 0)$ do not contain $x - \xi$ as a factor. Indeed, in the contrary case it would follow that the conic ϕ_2 and the cubic ϕ_3 degenerate into the line λ_1 and into residual components. The equation of the quartic curve (25) would be in this case

$$\lambda_1 \mu_1^3 + \lambda_2^2 = 0,$$

where $\mu_1 = 0$ is a line and $\lambda_2 = 0$ is a conic. But this equation represents a quartic with two cusps only.* We will call the representation (26) *non-degenerate,* if $x - \xi$ is not a factor of $\phi_2(x, 0)$ and $\phi_3(x, 0)$. It can be easily proved in a manner similar to that adopted by Clebsch for the general case, that if $f_4(x, 0)$ is given, then ξ may be taken arbitrarily, and once ξ is fixed, the problem of finding the non-degenerate representation (26) is connected with the problem of the trisection of the periods in the case of the elliptic curve

$$y^2 = f_4(x, 0).$$

There exist namely 4 different non-degenerate representations of the polynomial $f_4(x, 0)(x - \xi)^2$ as a sum of a square and a cube, each representation corresponding to two third-period characteristics $(\epsilon)_3$, $(-\epsilon)_3$.

We deduce that the three-cuspidal quartics which pass through 4 given points on a line belong to 4 different continuous systems. As in the preceding section we find the fundamental group of the quartic by starting from the relation $g_1 = g_2$, and by transforming it by those transformations of the loops which leave a conveniently chosen characteristic invariant, say the characteristic $(0; 1/3)$. We obtain thus the following generating relations:

(a) $g_1 = g_2$;

(b) $g_1 g_3 g_1 = g_3 g_1 g_3$;

(c) $g_1 g_4 g_1 = g_4 g_1 g_4$;

(d) $g_1^2 g_3 g_4 = 1$.

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* The statement in the text can be proved also by observing that if the cubic surface possesses a double line, the polar plane of a generic point of the space does not pass through the double line.

Substituting in (c) the expression of g_4, determined by (d), we find

$$g_1 g_3 g_1 g_3^{-1} g_1 g_3 = 1,$$

and, by (b), this relation becomes

$$g_1^2 g_3^2 = 1. \tag{e}$$

We have

$$(g_1 g_3)^3 = g_1 g_3 g_1 \cdot g_3 g_1 g_3 = g_1 g_3 g_1^2 g_3 g_1 = g_1^2,$$

and moreover,

$$(g_1 g_3)^3 = g_3 g_1 g_3 \cdot g_3 g_1 g_3 = g_3^2.$$

Hence $g_1^2 = g_3^2$, and, by (e), $g_1^4 = g_3^4 = 1$.
We find the following complete set of generating relations:

$$g_1^2 = g_3^2; \qquad g_1^4 = 1; \qquad (g_1 g_3)^3 = g_1^2. \tag{27}$$

The group defined by the relations (27) *is a finite group of order* 12. It is isomorphic with the *regular* transitive group of degree 12, generated by the following substitutions:

$$\begin{aligned} s_1 &= (z_1 z_2 z_3 z_4)(z_5 z_6 z_7 z_8)(z_9 z_{10} z_{11} z_{12}); \\ s_3 &= (z_1 z_5 z_3 z_7)(z_2 z_9 z_4 z_{11})(z_6 z_{10} z_8 z_{12}). \end{aligned} \tag{28}$$

The problem of finding all types of algebraic functions which can be constructed upon the three-cuspidal quartic as branch curve, is equivalent to the problem of finding all transitive groups of substitutions, which can be generated by two substitutions s_1 and s_3, satisfying the above relations (27). Since the abstract group G is of order 12, we may have, as possible groups of substitutions, outside the regular group of degree 12, mentioned above, groups of degrees 6, 4, 3 and 2. It is easily shown that the only possible group of degree 4 is here the cyclic group, generated by the substitution

$$s_1 = s_3 = (z_1 z_2 z_3 z_4).$$

We find a transitive *regular* group of degree 6, in a (2, 1) isomorphism with the abstract group G_{12}, by putting:

$$s_1 = (z_1 z_2)(z_3 z_4)(z_5 z_6); \qquad s_3 = (z_1 z_3)(z_2 z_5)(z_4 z_6). \tag{29}$$

Finally we obtain the total group of degree 3, by putting

$$s_1 = (z_1 z_2), \qquad s_3 = (z_1 z_3). \tag{30}$$

Hence, we have the following theorem:

Excluding the obvious cases of the four-valued and of the two-valued functions possessing a cyclic monodromie group, there are only three types of birationally distinct irreducible algebraic functions, which admit the three cuspidal quartic as branch curve, one of order 3 (*and this gives rise to the cubic surface with a double line*), *one of order* 6 (*which is defined by the resolvent of the cubic equation in the previous case*), *and finally one of order* 12. *The monodromie group of each of these functions is generated by two substitutions, indicated by the formulas* (30), (29) *and* (28) *respectively.*

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